

A Quillen model structure for Gray-categories

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Abstract

A Quillen model structure on the category **Gray-Cat** of **Gray**-categories is described, for which the weak equivalences are the triequivalences. It is shown to restrict to the full subcategory **Gray-Gpd** of **Gray**-groupoids. This is used to provide a functorial and model-theoretic proof of the unpublished theorem of Joyal and Tierney that **Gray**-groupoids model homotopy 3-types. The model structure on **Gray-Cat** is conjectured to be Quillen equivalent to a model structure on the category **Tricat** of tricategories and strict homomorphisms of tricategories.

Much of the recent development of higher category theory has been inspired and guided by the idea that higher groupoids should classify homotopy types. This is sometimes called the “homotopy hypothesis”. A higher groupoid is a higher category in which morphisms at all dimensions are invertible in a suitable sense; here, that will mean that they are strictly invertible, so that the composite of a morphism with its inverse is literally equal to an identity morphism. The goal of this paper is to describe certain precise relationships between higher categories and homotopy types, in low dimensions, using the machinery of model categories and Quillen equivalences.

Experience so far suggests that for each dimension n there should be a model category which captures what it is to “do” n -dimensional category theory, and that this should restrict to a model category on n -dimensional groupoids which provide a model for homotopy n -types.

The case $n = 1$ is well-understood. The category **Cat** of (small) categories and functors has a model structure [10] for which the weak equivalences are the equivalences of categories, and the fibrations are the functors with the isomorphism-lifting property. (The word fibration will always be used in the model-categorical sense; we shall not need to discuss categorical fibrations.) This model structure on **Cat** is sometimes given the epithet “folklore”, but I prefer to call it the *categorical model structure*, since it corresponds to what it is to do category theory. In category theory one regards two categories as being the same when they are equivalent (and then uses the equivalence to identify them). Furthermore the fibrations for the model structure arise naturally as the morphisms with the property that pulling back along them sends equivalences to equivalences; there is also an analogous characterizations of the cofibrations [9]. This model structure on **Cat** restricts to the full subcategory **Gpd** of **Cat** consisting of the groupoids. The nerve functor $N : \mathbf{Gpd} \rightarrow \mathbf{SSet}$ is the right adjoint part of a Quillen functor, and this allows us to regard groupoids as models for homotopy 1-types; this is made more precise in Section 6, where we see that $N : \mathbf{Gpd} \rightarrow \mathbf{SSet}$ is actually a Quillen equivalence when we replace the usual model structure on **SSet** with a localized version, which kills all homotopy information in dimension greater than 1. Thus groupoids are seen as the “intersection of category theory and homotopy theory”.¹

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¹We shall not have cause to consider the other well-known model structure on **Cat**, due to Thomason [19] — this is in fact Quillen equivalent to that of **SSet**, and so using this would give a rather different “intersection”. But while this Thomason

Similarly, when we turn to the case $n = 2$, the category **2-Cat** of (strict) 2-categories and (strict) 2-functors has a model structure [12, 13], for which the weak equivalences are the biequivalences, and the fibrations are 2-functors allowing liftings of both 1-cells which are equivalences and 2-cells which are invertible. Once again, these biequivalences capture exactly the usual notion of “sameness” for 2-categories, and the fibrations can be characterized in terms of pulling back of biequivalences. The model structure restricts to the full subcategory **2-Gpd** of **2-Cat** consisting of the 2-groupoids: these are the 2-categories in which all 1-cells and 2-cells are (strictly) invertible. The resulting model structure on **2-Gpd** was introduced in [17], prior to [12]. Once again there is a nerve functor $N : \mathbf{2-Gpd} \rightarrow \mathbf{SSet}$ which is the right adjoint part of a Quillen functor, and is in fact a Quillen equivalence for a localized model structure on **SSet**, obtained by killing homotopy information in dimension greater than 2; once again, this last aspect is described in more detail in Section 6.

The category **2-Cat** uses strict notions both for objects (2-categories) and morphisms (2-functors). One might question whether this is really suitable, given that in practice many 2-dimensional structures are not strict. The reason for using 2-functors is that they are much better behaved than the more general pseudofunctors, and that the more general ones are in any case encoded via the model structure. This is the typical situation: use well-behaved gadgets as models, then use the model structure to get at the more general notions. As far as the objects go, the standard notion of weak 2-category is called a bicategory, and the category **Bicat** of bicategories and strict homomorphisms of these also has a model structure [13], which is Quillen equivalent to that on **2-Cat**. This fact includes the result [15] that every bicategory is biequivalent to a strict one.

When we come to the case $n = 3$, the general notion of weak 3-category is called a tricategory, and this time it is not the case that every tricategory is suitably equivalent (“triequivalent”) to a strict one, however every tricategory is triequivalent to an intermediate structure called a Gray-category [4]. One of the main results proved below is that there is a model structure on the category **Gray-Cat** of Gray-categories for which the weak equivalences are the triequivalences. A Gray-groupoid is a Gray-category in which all 1-cells, 2-cells, and 3-cells have strict inverses. We shall also see that the model structure on **Gray-Cat** restricts to give one on the full subcategory **Gray-Gpd** of Gray-groupoids. There is a nerve functor $N : \mathbf{Gray-Gpd} \rightarrow \mathbf{SSet}$, defined in [1], which turns out to be the right adjoint part of a Quillen adjunction, and as in the previous cases this becomes a Quillen equivalence when we localize **SSet** by killing homotopies; this time those in dimension greater than 3.

This provides a model-theoretic and functorial formulation of the unpublished but widely advertised result of Joyal and Tierney that Gray-groupoids model homotopy 3-types. The possibility of such a formulation was suggested by Berger in [1]. In [1, Theorem 3.3], Berger considered the nerve functor $N : \mathbf{Gray-Gpd} \rightarrow \mathbf{SSet}$ and its left adjoint Π_3 , and asserted that the model structure on **SSet** could be transported across this adjunction to give a model structure on **Gray-Gpd**, and then show that the adjunction would induce an equivalence at the homotopy level between Gray-groupoids and simplicial 3-types. He did not, however, give a proof that the model structure actually transports. Once one knows that the structure transports, the fact that there is a Quillen adjunction is immediate, while the fact that is a Quillen equivalence, and so induces an equivalence of homotopy categories, essentially amounts to [1, Proposition 3.2].

I take a different approach, constructing the model structure on **Gray-Gpd** from that on **Gray-Cat**, rather than transporting it from **SSet**. The existence of the model structure is then more or less immediate, but one must show that the nerve is part of a Quillen adjunction — this contrasts with the approach of [1], where the work to be done is in constructing the model structure itself. The arguments given here to show that the nerve is part of a Quillen adjunction can in fact be used to complete the proof in [1] of the existence of the model structure. In either approach one then must do some work (essentially the same) to get a Quillen equivalence: the key step is [1, Proposition 3.2].

It turns out, after the fact, that the two model structures on **Gray-Gpd** actually agree: see Corollary 5.5.

Unlike the case $n = 2$, we do not yet have a model structure on the category of tricategories, but there is every reason believe that such a structure exists, and that it will be Quillen equivalent to our model structure

model structure is important in homotopy theory, it seems to correspond to another use of categories than category theory itself.

on **Gray-Cat**: we describe what is known in Section 7.

We now give a brief outline of the paper. Section 1 recalls some background material on enriched categories, on model categories, on 2-categories, and on Gray-categories. Section 2 describes the model structure, while Section 3 gives an alternative characterization of the fibrations, and Section 4 proves the remaining model category axioms. In Section 5, we look at the restricted model structure for Gray-groupoids, and its behaviour with respect to the nerve functor for Gray-groupoids. Section 6 shows how this relationship can be described using Bousfield localizations of the model category of simplicial sets. Section 7 describes briefly what is known about tricategories, and conjectures what else might be true. The last two sections concern the cofibrations for the model structure on **Gray-Cat** and in particular the cofibrant objects: Section 8 contains some preliminary material on computads, and Section 9 a characterization of cofibrations of Gray-categories and of cofibrant Gray-categories.

I first proved the existence of the model structure on **Gray-Cat** in January 2007. Since then, the paper has evolved in various ways, thanks in part to helpful conversations with a number of people. I received many helpful suggestions from Clemens Berger; among other things, when I explained to Clemens the model structure, he suggested that it might be possible to characterize the fibrations in the way given below, in terms of P_* and π_* . John Harper suggested the particular form that the localizations in Section 6 might take. I am grateful to both of them, and also to Michael Batanin and Richard Garner, for enlightening discussions on other aspects of the paper.

1 Background

1.1 Enriched categories

For a monoidal category $\mathcal{V} = (\mathcal{V}, \otimes, I)$, we write $\mathcal{V}\text{-Cat}$ for the category of \mathcal{V} -categories and \mathcal{V} -functors; this can be made into a 2-category with the \mathcal{V} -natural transformations as 2-cells, but we shall not need to do so. The most important case will be where \mathcal{V} is the category **2-Cat** of 2-categories and 2-functors, and \otimes is what is called the Gray tensor product; then the \mathcal{V} -categories are precisely the Gray-categories of the introduction. When \mathcal{V} is the cartesian closed category **Cat**, a \mathcal{V} -category is a 2-category. If A and B are objects of a \mathcal{V} -category \mathcal{A} , we write $\mathcal{A}(A, B)$ for the corresponding hom-object in \mathcal{V} .

For monoidal categories $\mathcal{V} = (\mathcal{V}, \otimes, I)$ and $\mathcal{W} = (\mathcal{W}, \otimes, I)$, a monoidal functor $P : \mathcal{V} \rightarrow \mathcal{W}$ consists of a functor P between the underlying categories, equipped with natural coherent morphisms $PX \otimes PY \rightarrow P(X \otimes Y)$ and $I \rightarrow PI$. Such a monoidal functor induces a 2-functor $P_* : \mathcal{V}\text{-Cat} \rightarrow \mathcal{W}\text{-Cat}$ sending a \mathcal{V} -category \mathcal{A} to the \mathcal{W} -category $P_*\mathcal{A}$ with the same objects as \mathcal{A} , but with \mathcal{W} -valued homs given by $(P_*\mathcal{A})(A, B) = P(\mathcal{A}(A, B))$. In particular, for any monoidal category \mathcal{V} the representable functor $\mathcal{V}(I, -) : \mathcal{V} \rightarrow \mathbf{Set}$ arising from the unit I is monoidal, and so induces a 2-functor $\mathcal{V}\text{-Cat} \rightarrow \mathbf{Cat}$ sending a \mathcal{V} -category \mathcal{A} to its underlying ordinary category \mathcal{A}_0 .

If \mathcal{V} is a monoidal category, a \mathcal{V} -category \mathcal{A} is said to have a property “locally” if each hom-object $\mathcal{A}(A, B)$ has the property (as an object of \mathcal{V}). Similarly a \mathcal{V} -functor $F : \mathcal{A} \rightarrow \mathcal{B}$ has a property if each $F : \mathcal{A}(A, B) \rightarrow \mathcal{B}(FA, FB)$ has the property (as a morphism of \mathcal{V}). So for example, one might call a \mathcal{V} -functor $F : \mathcal{A} \rightarrow \mathcal{B}$ “locally an isomorphism” if each $F : \mathcal{A}(A, B) \rightarrow \mathcal{B}(FA, FB)$ is an isomorphism in \mathcal{V} ; more commonly, however, such an F is called fully faithful.

A \mathcal{V} -functor $F : \mathcal{A} \rightarrow \mathcal{B}$ is an equivalence if there exists a \mathcal{V} -functor $G : \mathcal{B} \rightarrow \mathcal{A}$ with $GF \cong 1$ and $FG \cong 1$. This is the case if and only if F is fully faithful (in the sense of the previous paragraph) and essentially surjective on objects (for each $B \in \mathcal{B}$ there is an $A \in \mathcal{A}$ with $B \cong FA$ in \mathcal{B}). Note that an isomorphism $B \cong FA$ in \mathcal{B} is the same thing as an isomorphism in the underlying ordinary category \mathcal{B}_0 of \mathcal{B} .

1.2 Monoidal model categories

A *monoidal model category* [8] is a category that has both a symmetric monoidal closed structure and a model structure, satisfying some compatibility conditions: the “pushout product” axiom should hold, as

should a condition on the tensor unit; the latter condition is automatic if, as in our examples, the unit is cofibrant.

All that is really needed in this paper is that if \mathcal{V} is a monoidal model category, then the homotopy category $\mathrm{ho}\mathcal{V}$ of \mathcal{V} inherits a derived monoidal structure for which the canonical functor $P : \mathcal{V} \rightarrow \mathrm{ho}\mathcal{V}$ is strong monoidal. We shall also write $\pi : \mathcal{V} \rightarrow \mathbf{Set}$ for the composite of P with the monoidal functor $\mathrm{ho}\mathcal{V}(I, -) : \mathrm{ho}\mathcal{V} \rightarrow \mathbf{Set}$ given by homming out of the unit. This also induces a functor $\pi_* : \mathcal{V}\text{-}\mathbf{Cat} \rightarrow \mathbf{Cat}$.

1.3 2-categories

In this section we recall a few basic ideas about 2-categories. Adjunctions can be defined in any 2-category, with the usual notion of adjunction being the case of the 2-category \mathbf{Cat} . An adjunction in a 2-category \mathcal{K} consists of morphisms $f : A \rightarrow B$ and $g : B \rightarrow A$ with 2-cells $\eta : 1 \rightarrow gf$ and $\epsilon : gf \rightarrow 1$ satisfying the triangle equations.

Such an adjunction is called an *adjoint equivalence* when the unit η and counit ϵ are both invertible. Occasionally I may allow myself to say that a morphism f “is an adjoint equivalence”; this always means that some definite choice of g , η , and ϵ has been made.

On the other hand, a morphism f is an *equivalence* when there *exists* a morphism g with $gf \cong 1$ and $fg \cong 1$. As is well-known (see [12] for example), if f is an equivalence and $\eta : 1 \cong gf$ is an isomorphism, then there is exactly one choice of ϵ for which the one (either) triangle equation holds; then the other will also hold, and we shall have an adjoint equivalence. (Similarly given ϵ there is exactly one way to choose η .)

The category $\mathbf{2-Cat}$ is locally finitely presentable, by [11], and so small object arguments work smoothly. We shall consider $\mathbf{2-Cat}$ with the model structure of [12]. The corresponding homotopy category has 2-categories as objects, and pseudonatural equivalence classes of pseudofunctors as morphisms. A weak equivalence in $\mathbf{2-Cat}$ is (a 2-functor which is) a biequivalence $F : \mathcal{A} \rightarrow \mathcal{B}$: this means that F is locally an equivalence, so that each $F : \mathcal{A}(A, B) \rightarrow \mathcal{B}(FA, FB)$ is an equivalence; and furthermore that F is biessentially surjective, so that each object $X \in \mathcal{B}$ is equivalent to one of the form FA for some $A \in \mathcal{A}$.

1.4 Gray-categories

We shall primarily be interested in the case where \mathcal{V} is the category $\mathbf{2-Cat}$, equipped with the model structure of [12] and the symmetric monoidal closed structure given by the Gray tensor product [5]; this monoidal category is often called **Gray**, to distinguish it from the monoidal category with the cartesian product as tensor product.

A 3-category is a category enriched in $\mathbf{2-Cat}$ with the cartesian product; a **Gray**-category is a category enriched in $\mathbf{2-Cat}$ with the Gray tensor product. The difference between **Gray**-categories and 3-categories is that while given 2-cells

$$\begin{array}{ccccc} & f & & g & \\ A & \xrightarrow{\quad} & B & \xrightarrow{\quad} & C \\ & \Downarrow \alpha & & \Downarrow \beta & \\ & f' & & g' & \end{array}$$

in a 3-category, the composites $\beta f'.g\alpha$ and $g'\alpha.\beta f$ agree, in a **Gray**-category they need not; rather, there is a specified isomorphism which we call simply $\beta\alpha$ or β_α . These are sometimes called the *pseudonaturality isomorphisms*.

A morphism $f : A \rightarrow B$ in a **Gray**-category \mathbb{A} is called a *biequivalence in \mathbb{A}* if there is a morphism $g : B \rightarrow A$ with $gf \simeq 1$ and $fg \simeq 1$. For any **Gray**-functor $F : \mathbb{A} \rightarrow \mathbb{B}$ we have $Fg.Ff = F(gf) \simeq F1 = 1$ and $Ff.Fg \simeq 1$, and so Ff is also a biequivalence.

Then $\mathrm{ho}\mathbf{Gray}$ is the category of 2-categories and equivalence classes of pseudofunctors, while $\pi : \mathbf{Gray} \rightarrow \mathbf{Set}$ sends a 2-category \mathcal{A} to the set of equivalence classes of objects of \mathcal{A} (in other words objects A and B are equivalent exactly when they are equivalent in the 2-category \mathcal{A}). We shall write **Gray-Cat** for the category of **Gray**-categories and **Gray**-functors (the 2-cells will have little explicit role). The category $\mathbf{2-Cat}$ is locally finitely presentable, thus so by [11] is **Gray-Cat**. This means that small object arguments work smoothly in **Gray-Cat** (all objects are small).

We shall frequently use the functors $P_* : \mathbf{Gray-Cat} \rightarrow \mathbf{ho Gray-Cat}$ and $\pi_* : \mathbf{Gray-Cat} \rightarrow \mathbf{Cat}$.

2 The model structure on Gray-Cat

We define a **Gray**-functor $F : \mathbb{A} \rightarrow \mathbb{B}$ to be a *weak equivalence* if it induces an equivalence $P_*F : P_*\mathbb{A} \rightarrow P_*\mathbb{B}$ of \mathbf{hoGray} -categories. We shall spell out more explicitly what this means in the following paragraphs, but it is immediate from this definition that these weak equivalences are closed under retracts and satisfy the 2-out-of-3 property, since equivalences of enriched categories have these closure properties.

To say that P_*F is an equivalence is to say that it is fully faithful and essentially surjective on objects. Being fully faithful means that each $P_*F : P_*\mathbb{A}(A, B) \rightarrow P_*\mathbb{B}(FA, FB)$ is invertible in \mathbf{hoGray} ; in other words, each $PF : P(\mathbb{A}(A, B)) \rightarrow P(\mathbb{B}(FA, FB))$ is invertible in \mathbf{hoGray} ; but this is just to say that each $F : \mathbb{A}(A, B) \rightarrow \mathbb{B}(FA, FB)$ is a weak equivalence in **Gray**; in other words, a biequivalence.

On the other hand P_*F is essentially surjective when for every $B \in \mathbb{B}$ there is an $A \in \mathbb{A}$ with an isomorphism $B \cong FA$ in $P_*\mathbb{B}$. But isomorphisms in $P_*\mathbb{B}$ are just isomorphisms in the underlying ordinary category $\pi_*\mathbb{B}$ of $P_*\mathbb{B}$, so this is just saying that $\pi_*F : \pi_*\mathbb{A} \rightarrow \pi_*\mathbb{B}$ is essentially surjective on objects.

We can also make this still more explicit. The category $\pi_*\mathbb{B}$ has the same objects as \mathbb{B} , but a morphism in $\pi_*\mathbb{B}$ from B to C is an equivalence class of pseudofunctors $1 \rightarrow \mathbb{B}(B, C)$. So such a morphism can be represented by a morphism $f : B \rightarrow C$ in \mathbb{B} , but two such $f, g : B \rightarrow C$ represent the same morphism in $(\mathbf{ho}_*\mathbb{B})_0$ if and only if they are equivalent. So objects B and C of $\mathbf{ho}_*\mathbb{B}$ are isomorphic if and only if there exist $f : B \rightarrow C$ and $g : C \rightarrow B$ with equivalences $gf \simeq 1$ and $fg \simeq 1$; that is, if and only if they are *biequivalent* in \mathbb{B} . And this means that the weak equivalences in **Gray-Cat** are precisely the *triequivalences* of **Gray**-categories.

We define a **Gray**-functor $F : \mathbb{A} \rightarrow \mathbb{B}$ to be a *fibration* if it induces fibrations $F : \mathbb{A}(A, B) \rightarrow \mathbb{B}(FA, FB)$ in \mathcal{V} between the hom-objects, for all $A, B \in \mathbb{A}$, and if moreover the functor $\pi_*F : \pi_*\mathbb{A} \rightarrow \pi_*\mathbb{B}$ is a fibration in **Cat** (an isofibration). This means that for every object $A \in \mathbb{A}$ and every isomorphism $f : B \cong FA$ in $\pi_*\mathbb{B}$, there is an object $A' \in \mathbb{A}$ with $FA' = B$ and an isomorphism $f' : A' \cong A$ in $\pi_*\mathbb{A}$ with $Ff' = f$ (in $\pi_*\mathbb{B}$).

Of course a **Gray**-functor $F : \mathbb{A} \rightarrow \mathbb{B}$ is a *trivial fibration* if it is weak equivalence and a fibration; this means that each $F : \mathbb{A}(A, B) \rightarrow \mathbb{B}(FA, FB)$ is a weak equivalence and a fibration — that is, a trivial fibration in **2-Cat**, and that each $\pi_*F : \pi_*\mathbb{A} \rightarrow \pi_*\mathbb{B}$ is essentially surjective on objects and an isofibration. This certainly implies that π_*F is surjective on objects; but π_* does not affect the objects, and so F itself is surjective on objects. We can now prove:

Proposition 2.1 *A Gray-functor $F : \mathbb{A} \rightarrow \mathbb{B}$ is a trivial fibration if and only if it is surjective on objects and is locally a trivial fibration in 2-Cat.*

PROOF: We have already seen the “only if” part. For the converse, suppose that F is surjective on objects and a trivial fibration on the homs. Then certainly it is a weak equivalence, and is a fibration on the homs. It remains to show that π_*F is an isofibration. But since F is a weak equivalence on the homs, π_*F is fully faithful; since it is also surjective on objects, it is a trivial fibration in **Cat**, and so in particular an isofibration. \square

We define the cofibrations to be the **Gray**-functors with the left lifting property with respect to the trivial fibrations. For a 2-category X we define $\mathbf{2}_X$ to be the **Gray**-category with two objects 0 and 1, with $\mathbf{2}_X(0, 0) = \mathbf{2}_X(1, 1) = 1$, $\mathbf{2}_X(0, 1) = X$, and $\mathbf{2}_X(1, 0) = 0$. This is clearly functorial in X , and for any **Gray**-category \mathbb{A} , there is a bijection between **Gray**-functors $\mathbf{2}_X \rightarrow \mathbb{A}$ and pairs of objects $A, B \rightarrow \mathbb{A}$ along with a 2-functor $X \rightarrow \mathbb{A}(A, B)$. We may now take as generating cofibrations the $\mathbf{2}_j : \mathbf{2}_X \rightarrow \mathbf{2}_Y$ for each generating cofibration $j : X \rightarrow Y$ in **2-Cat**, and the unique **Gray**-functor $0 \rightarrow 1$; and the cofibrations and trivial fibrations from a cofibrantly generated weak factorization system with respect to these generating cofibrations.

We shall prove the following theorem:

Theorem 2.2 *This choice of cofibrations, fibrations, and weak equivalences makes **Gray-Cat** into a (combinatorial) model category.*

The key step is to show that fibrations can be defined via a right lifting property. We shall do this in the next section. Given that, the remainder of the proof is largely formal.

Remark 2.3 For any monoidal model category \mathcal{V} we could define a \mathcal{V} -functor $F : \mathbb{A} \rightarrow \mathbb{B}$ to be a weak equivalence if $P_*F : P_*\mathbb{A} \rightarrow P_*\mathbb{B}$ is an equivalence of $\text{ho}\mathcal{V}$ -categories, and to be a fibration if F is locally a fibration in \mathcal{V} and $\pi_*F : \pi_*\mathbb{A} \rightarrow \pi_*\mathbb{B}$ is a fibration in **Cat**. Then the trivial fibrations would be precisely the \mathcal{V} -functors which are surjective on objects and locally a trivial fibration in \mathcal{V} , and the cofibrations could be defined to be the morphisms with the left lifting property with respect to the trivial fibrations. In order to prove the model category axioms, however, we shall use various properties of the monoidal model category $\mathcal{V} = \mathbf{Gray}$.

3 Fibrations of Gray-categories and adjoint biequivalences in Gray-categories

Recall that a **Gray**-functor $F : \mathbb{A} \rightarrow \mathbb{B}$ is a fibration if it satisfies the following two conditions:

- (i) The 2-functor $F : \mathbb{A}(A, B) \rightarrow \mathbb{B}(FA, FB)$ is a fibration in **2-Cat** for all objects $A, B \in \mathbb{A}$
- (ii) For each $A \in \mathbb{A}$ and each isomorphism $f : B \cong FA$ in $\pi_*\mathbb{B}$, there is an $A' \in \mathbb{A}$ with $FA' = B$ and an isomorphism $f' : A' \cong A$ in $\pi_*\mathbb{A}$ with $Ff' = f$.

The first condition clearly amounts to the fact that F has the right lifting property with respect to the **Gray**-functors $2_j : 2_X \rightarrow 2_Y$ for each generating trivial cofibration $j : X \rightarrow Y$ of **2-Cat**. The second condition is not, as it stands, a right lifting property of F , since it involves $\pi_*\mathbb{A}$ and $\pi_*\mathbb{B}$. We shall show how to replace condition (ii) with a condition (ii*) which is a right lifting property, and which, in the presence of (i), is equivalent to (ii).

Recall that a morphism $f : A \rightarrow B$ in a **Gray**-category \mathbb{B} is a biequivalence if there exist a $g : B \rightarrow A$ with $1 \simeq gf$ and $fg \simeq 1$. We define, following [20], an *adjoint biequivalence* in \mathbb{B} to consist of $f : A \rightarrow B$ and $g : B \rightarrow A$, equipped with adjoint equivalences $\eta : 1 \simeq gf$ and $\epsilon : fg \simeq 1$, and isomorphisms $S : \epsilon f \cdot f \eta \cong 1$ and $T : 1 \cong g \epsilon \cdot \eta g$ for which the pasting composites

are identities. One might call these last conditions the *tetrahedron equations*. Clearly f and g are then biequivalences. Note that we have simply said that $\eta : 1 \simeq gf$ and $\epsilon : fg \simeq 1$ are adjoint equivalences, but that this must be understood to mean that we have chosen all the remaining structure of an adjoint equivalence (e.g. a morphism $\eta^* : gf \rightarrow 1$ and invertible 2-cells $\eta^*\eta \cong 1$ and $\eta\eta^* \cong 1$ satisfying the triangle equations).

We shall show that every biequivalence can be made into an adjoint biequivalence. More precisely:

Proposition 3.1 *Let $f : A \rightarrow B$ be a biequivalence.*

- (i) If $g : B \rightarrow A$, and $\epsilon : fg \simeq 1$ is an equivalence, then there exists an equivalence $\eta : 1 \simeq gf$ and an isomorphism $S : \epsilon f.f\eta \cong 1$.
- (ii) If $g : B \rightarrow A$, if $\eta : 1 \simeq gf$ and $\epsilon : fg \simeq 1$ are equivalences, and if $S : \epsilon f.f\eta \cong 1$ is an isomorphism, then there exists a unique isomorphism $T : g\epsilon.\eta g \cong 1$ satisfying the first tetrahedron equation.
- (iii) If $g : B \rightarrow A$, if $\eta : 1 \simeq gf$ and $\epsilon : fg \simeq 1$ are equivalences, if $S : \epsilon f.f\eta \cong 1$ and $T : g\epsilon.\eta g \cong 1$ are isomorphisms, and if the first tetrahedron equation holds then so does the second.

PROOF: (i) The 2-functor $\mathbb{A}(A, f) : \mathbb{A}(A, A) \rightarrow \mathbb{A}(A, B)$ is a biequivalence, and so locally an equivalence; in particular the induced functor

$$\mathbb{A}(A, A)(1, gf) \xrightarrow{\mathbb{A}(A, f)} \mathbb{A}(A, B)(f, fgf)$$

is an equivalence and so

$$\mathbb{A}(A, A)(1, gf) \xrightarrow{\mathbb{A}(A, f)} \mathbb{A}(A, B)(f, fgf) \xrightarrow{\mathbb{A}(f, \epsilon f)} \mathbb{A}(A, B)(f, f)$$

is an equivalence, since ϵ is one. So by essential surjectivity, there is an $\eta : 1 \rightarrow gf$ whose image $\epsilon f.f\eta$ is isomorphic to the identity on f , say by $S : \epsilon f.f\eta \cong 1$. It remains to check that η is an equivalence. Now ϵ is an equivalence, so ϵf is an equivalence, so $f\eta$ is an equivalence; but f is a biequivalence, and thus η is indeed an equivalence.

(ii) The 2-functor $\mathbb{A}(B, f) : \mathbb{A}(B, A) \rightarrow \mathbb{A}(B, B)$ is a biequivalence, and so locally an equivalence; in particular the induced functor

$$\mathbb{A}(B, A)(g, g) \xrightarrow{\mathbb{A}(B, f)} \mathbb{A}(B, B)(fg, fg)$$

is an equivalence and so

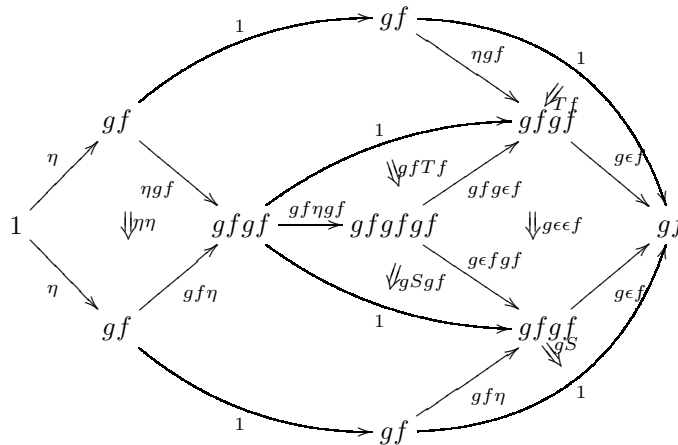
$$\mathbb{A}(B, A)(g, g) \xrightarrow{\mathbb{A}(B, f)} \mathbb{A}(B, B)(fg, fg) \xrightarrow{\mathbb{A}(B, B)(fg, \epsilon)} \mathbb{A}(B, B)(fg, 1) \quad (1)$$

is an equivalence. Now this composite equivalence maps the identity on g to ϵ , and maps $g\epsilon.\eta g$ to $\epsilon.fg\epsilon.f\eta g$, but we have isomorphisms

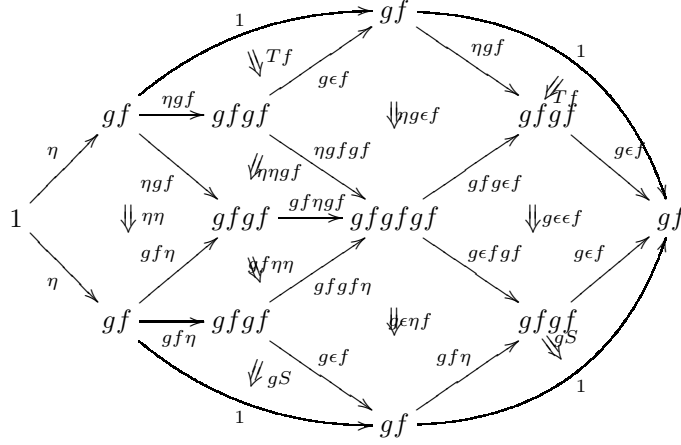
$$\epsilon.fg\epsilon.f\eta g \xrightarrow{\epsilon_\epsilon f\eta g} \epsilon.\epsilon fg.f\eta g \xrightarrow{\epsilon.Sg} \epsilon \quad (2)$$

in $\mathbb{A}(B, B)(fg, 1)$ and so a unique isomorphism $T : g\epsilon.\eta g \cong 1$ sent by (1) to (2); that is, a unique T satisfying the first tetrahedron equation.

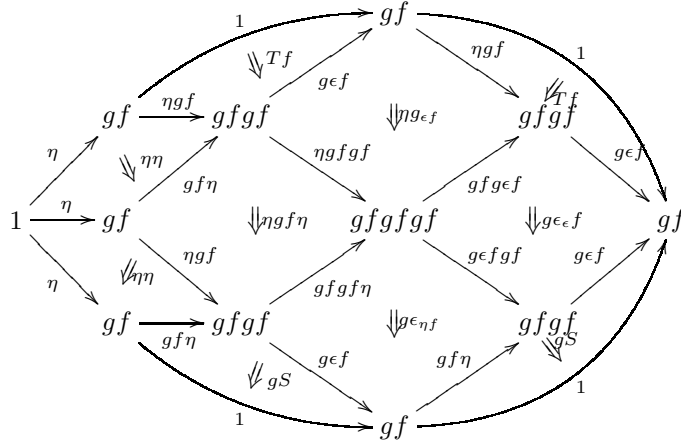
(iii) Consider the diagram



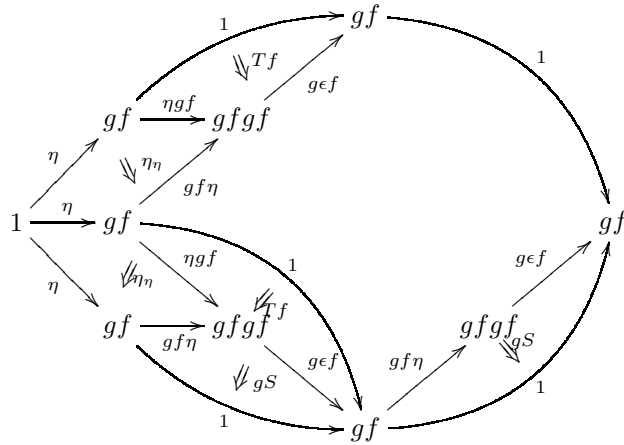
We are to show that the second tetrahedron equation holds. Since g and f are biequivalences, it will suffice to show that the composite of the central 3 cells above is an identity; but since the first tetrahedron equation holds, this is equivalent to the whole displayed diagram being an identity. By naturality of $\eta g f$ and $g f \eta$, this is equal to



which by naturality of the pseudonaturality isomorphisms is equal to



and now by naturality of T with respect to $g \epsilon f . g f \eta$ this is



which by two applications of the first tetrahedron equation is the identity. \square

Remark 3.2 In fact given (η, S) as in (i), and another choice (η', S') , there is a unique invertible $Y : \eta \cong \eta'$ compatible with S and S' . Similarly, given (g', ϵ') in place of (g, ϵ) , there is a suitable equivalence $g \simeq g'$. In summary, the “space of ways of making f into an adjoint biequivalence” is contractible. We shall not need this fact, and so do not bother to formulate it precisely.

We are almost ready to give our condition (ii*). First we record the following easy result:

Lemma 3.3 *Let $P : \mathcal{E} \rightarrow \mathcal{B}$ be a fibration in **2-Cat**, and $f : D \rightarrow E$ an equivalence in \mathcal{E} . If $\beta : g \cong Pf$ is an invertible 2-cell, and g is part of an adjoint equivalence $(g : PD \rightarrow PE, g^* : PE \rightarrow PD, \eta : 1 \cong g^*g, \epsilon : gg^* \cong 1)$, then for any invertible lifting $\bar{\beta} : \bar{g} \cong f$ of β , we can make \bar{g} into an adjoint equivalence $(\bar{g}, \bar{g}^*, \eta', \epsilon')$ over (g, g^*, η, ϵ) .*

PROOF: First make the equivalence f into an adjoint equivalence $(f, f_1, \eta_1, \epsilon_1)$. Then $(Pf, Pf_1, P\eta_1, P\epsilon_1)$ is an adjoint equivalence in \mathbb{B} . The isomorphism $\beta : g \cong Pf$ determines a unique isomorphism $\beta^* : g^* \cong Pf_1$ making the diagram

$$\begin{array}{ccc} 1 & \xrightarrow{\eta} & g^*g \\ P\eta_1 \downarrow & & \downarrow g^*\beta \\ Pf_1.Pf & \xrightarrow{P(\beta^*)^{-1}.Pf} & g^*.Pf \end{array}$$

commute. Lift $\beta^* : g^* \cong Pf_1$ to an isomorphism $\bar{\beta}^* : \bar{g}^* \cong f_1$ over β^* . Let $\eta' : 1 \rightarrow \bar{g}^*\bar{g}$ be the isomorphism

$$1 \xrightarrow{\eta_1} f_1f \xrightarrow{\bar{\beta}^{*-1}.f} \bar{g}^*.f \xrightarrow{\bar{g}^*.\bar{\beta}^{-1}} \bar{g}^*.\bar{g}$$

and now observe that $P\eta' = \eta$ by the defining property of β^* .

Now \bar{g} is isomorphic to the equivalence f , and so is itself an equivalence. We have an isomorphism $\eta' : 1 \cong \bar{g}^*\bar{g}$, and so there is a unique isomorphism $\epsilon' : \bar{g}\bar{g}^* \cong 1$ for which $(\bar{g}, \bar{g}^*, \eta', \epsilon')$ is an adjoint equivalence in \mathbb{E} . But then $(P\bar{g}, P\bar{g}^*, P\eta', P\epsilon')$ and (g, g^*, η, ϵ) are both adjoint equivalences in \mathbb{B} , with $P\bar{g} = g$, $P\bar{g}^* = g^*$, and $P\eta' = \eta$; so finally $P\epsilon' = \epsilon$, since the unit of an adjunction determines the counit. \square

Our condition (ii*) is now that $F : \mathbb{A} \rightarrow \mathbb{B}$ has the *adjoint-biequivalence-lifting property*: given $E \in \mathbb{E}$, $B \in \mathbb{B}$, $f : B \rightarrow FE$ and $g : FE \rightarrow B$ along with η, ϵ, S , and T giving an adjoint biequivalence in \mathbb{B} , there exist $D \in \mathbb{E}$ and $f' : D \rightarrow E$, $g' : E \rightarrow D$, $\eta' : 1 \rightarrow g'f'$, $\epsilon' : f'g' \rightarrow 1$, $S' : \epsilon'f'.f'\eta' \rightarrow 1$, $T' : 1 \rightarrow g'\epsilon'.\eta'g'$ forming an adjoint biequivalence in \mathbb{E} , with $Ff' = f$, $Fg' = g$, $F\eta' = \eta$, $F\epsilon' = \epsilon$, $FS' = S$, and $FT' = T$.

Any **Gray**-functor $F : \mathbb{A} \rightarrow \mathbb{B}$ which is a fibration on the homs and has the adjoint-biequivalence-lifting property is certainly a fibration: for given $A \in \mathbb{A}$ and an isomorphism $B \cong FA$ in $\pi_*\mathbb{B}$, we can represent the isomorphism by a biequivalence $f : B \rightarrow FA$ in \mathbb{B} , and then extend this to an adjoint biequivalence (involving g, η, ϵ , and so on), which by assumption can be lifted to an adjoint biequivalence involving $f' : A' \rightarrow A$ and other data, lying above the original adjoint biequivalence. In particular f' represents an isomorphism $A' \cong A$ in $\pi_*\mathbb{A}$ lying over the original isomorphism $B \cong FA$ in $\pi_*\mathbb{B}$. The harder part is:

Proposition 3.4 *Every fibration satisfies the adjoint-biequivalence-lifting property.*

PROOF: Let $P : \mathbb{E} \rightarrow \mathbb{B}$ be a fibration. Let $E, B, f, g, \eta, \epsilon, S$, and T be an adjoint biequivalence as above. Then f determines an isomorphism $B \cong FE$ in $\pi_*\mathbb{B}$, which can be lifted to an isomorphism $D \cong E$ in $\pi_*\mathbb{E}$, and this isomorphism can be represented by a biequivalence $f_0 : D \rightarrow E$ in \mathbb{E} , with $PD = B$ and Pf_0 equivalent in $\mathbb{B}(B, PE)$ to f . Since $P : \mathbb{E}(D, E) \rightarrow \mathbb{B}(B, PE)$ is a fibration in **2-Cat**, we can lift the equivalence $Pf_0 \simeq f$ to an equivalence $f_0 \simeq f'$, so that $Pf' = f$; also f' is equivalent to the biequivalence f_0

so is itself a biequivalence. Thus we have lifted the biequivalence $f : B \rightarrow PE$ to a biequivalence $f' : D \rightarrow E$; we must now lift the remaining components of the adjoint biequivalence.

By Proposition 3.1, we can make f' into an adjoint biequivalence in \mathbb{E} via $(g_1 : E \rightarrow D, \eta_1, \epsilon_1, S_1, T_1)$, and now its image $(f, Pg_1, P\eta_1, P\epsilon_1, PS_1, PT_1)$ under P is an adjoint biequivalence in \mathbb{B} . But $(f, g, \eta, \epsilon, S, T)$ is also an adjoint biequivalence, so the composite

$$g \xrightarrow{P\eta_1 \cdot g} Pg_1 \cdot f \cdot g \xrightarrow{Pg_1 \cdot \epsilon} Pg_1$$

is (part of) an adjoint equivalence, and so can be lifted to an adjoint equivalence $\alpha : g' \rightarrow g_1$ in $\mathbb{E}(D, E)$, which in turn gives an adjoint equivalence

$$f'g' \xrightarrow{f'\alpha} f'g_1 \xrightarrow{\epsilon_1} 1$$

lying over the top leg of

$$\begin{array}{ccccc} Pf' \cdot Pg' & & & & \\ \parallel & \searrow^{Pf' \cdot P\alpha} & & & \\ fg & \xrightarrow{f \cdot P\eta' \cdot g} & f \cdot Pg_1 \cdot f \cdot g & \xrightarrow{f \cdot Pg_1 \cdot \epsilon} & f \cdot Pg_1 \\ & \searrow^{P\epsilon \cdot fg} & \downarrow & \searrow & \downarrow P\epsilon_1 \\ & & fg & \xrightarrow{\epsilon} & 1 \end{array}$$

1

in which the unnamed 2-cell is a pseudonaturality isomorphism. The pasting composite here is an invertible 2-cell in $\mathbb{B}(B, B)$ between adjoint equivalences, so can be lifted to an invertible 2-cell

$$\begin{array}{ccc} f'g' & \xrightarrow{f'\alpha} & f'g_1 \xrightarrow{\epsilon_1} 1 \\ & \searrow & \downarrow X \\ & & \epsilon' \end{array}$$

in $\mathbb{E}(D, D)$, and now the adjoint equivalence structure of $\epsilon_1 \cdot f'\alpha$ transports across the isomorphism X to give an adjoint equivalence structure on ϵ' , lying over that on $\epsilon : fg \rightarrow 1$, as in Lemma 3.3.

At this point we have $f' : D \rightarrow E$ and $g' : E \rightarrow D$ over f and g , and an adjoint equivalence $\epsilon' : f'g' \rightarrow 1$ over $\epsilon : fg \rightarrow 1$. Now f' is a biequivalence, so by Proposition 3.1 there exist an equivalence $\beta : 1 \rightarrow g'f'$ and an invertible 2-cell

$$\begin{array}{ccc} f' & \xrightarrow{f'\beta} & f'g'f' \\ & \searrow & \downarrow \epsilon' f' \\ & & f' \end{array}$$

1

which P sends to the left hand side of the diagram

$$\begin{array}{ccc} f & \xrightarrow{f \cdot P\beta} & fgf \\ & \searrow & \downarrow \epsilon f \\ & & f \end{array} \quad \begin{array}{ccc} f & \xrightarrow{f\eta} & fgf \\ & \searrow & \downarrow \epsilon f \\ & & f \end{array}$$

1

but since we also have the diagram on the right, and ϵ is an equivalence and f a biequivalence, there exists a unique invertible

$$\begin{array}{ccc} & \eta & \\ 1 & \Downarrow R & gf \\ & P\beta & \end{array}$$

which when pasted to PS_2 gives S . We can now lift this by Lemma 3.3 to

$$1 \begin{array}{c} \xrightarrow{\eta'} \\ \Downarrow R' \\ \xrightarrow{\beta} \end{array} g' f'$$

with η' an adjoint equivalence over η , and define S' to be the composite

$$\begin{array}{ccc} f' & \xrightarrow{f' \eta'} & f' g' f' \\ & \Downarrow f' R' & \downarrow \epsilon' f' \\ f' & \xrightarrow{f' \beta} & f' \\ & \Downarrow S_2 & \\ & 1 & \end{array}$$

and observe that this lies over S .

By Proposition 3.1 once again there is a unique invertible $T' : g' \epsilon' . \eta' g' \cong 1$ for which $(f', g', \eta', \epsilon', S', T')$ is an adjoint biequivalence; thus $(Pf' = f, Pg' = g, P\eta' = \eta, P\epsilon' = \epsilon, PS' = S, PT' = T)$ is an adjoint biequivalence, and by the uniqueness part of the last clause of Proposition 3.1, it follows that $PT' = T$. \square

This now proves:

Proposition 3.5 *A **Gray**-functor $F : \mathbb{A} \rightarrow \mathbb{B}$ is a fibration if and only if it satisfies the adjoint-biequivalence-lifting property and each $F : \mathbb{A}(A, B) \rightarrow \mathbb{B}(FA, FB)$ is a fibration in **2-Cat**.*

We have already observed that the condition on the homs is a right lifting property. For the adjoint biequivalence lifting property, note that the structure of adjoint biequivalence can be described in terms of objects, 1-cells, 2-cells, and 3-cells of \mathbb{A} satisfying certain equations; thus since **Gray-Cat** is locally finitely presentable, there is a “universal adjoint biequivalence”: a **Gray**-category \mathbb{E} , with the property that adjoint biequivalences in \mathbb{A} are in natural bijection with **Gray**-functors from \mathbb{E} to \mathbb{A} . There are two objects of \mathbb{E} ; the adjoint-biequivalence-lifting property is exactly the right lifting property with respect to one (either) of the **Gray**-functors $1 \rightarrow \mathbb{E}$. By the small object argument we conclude:

Proposition 3.6 *There is a cofibrantly generated weak factorization system on **Gray-Cat** whose right part is the fibrations. (The left part is what we call the trivial cofibrations.)*

Remark 3.7 Since all objects in **2-Cat** are fibrant, and since there are no non-trivial biequivalences in the terminal **Gray**-category, it is clear that all objects in **Gray-Cat** are also fibrant.

4 Proof of the model structure

It remains to show that the trivial cofibrations are precisely those cofibrations which are also weak equivalences. Since every trivial fibration is a fibration, certainly every trivial cofibration is a cofibration. We shall show that every trivial cofibration is a weak equivalence, and then use a standard argument to show that also every weak equivalence which is a cofibration is a trivial cofibration.

The key step in this proof is the existence of path objects:

Proposition 4.1 *Path objects exist in **Gray-Cat**: for every **Gray**-category \mathbb{B} there exists a **Gray**-category \mathbb{PB} and a factorization*

$$\mathbb{B} \xrightarrow{D} \mathbb{PB} \xrightarrow{\left(\begin{smallmatrix} P \\ P' \end{smallmatrix}\right)} \mathbb{B} \times \mathbb{B}$$

of the diagonal into a weak equivalence followed by a fibration.

PROOF: Define a **Gray**-category \mathbb{PB} as follows:

- an object is a biequivalence $a : A \rightarrow A'$ in \mathbb{B}
- a 1-cell from $a : A \rightarrow A'$ to $b : B \rightarrow B'$ consists of 1-cells $f : A \rightarrow B$ and $f' : A' \rightarrow B'$ in \mathbb{B} , and an equivalence $\phi : bf \simeq f'a$
- a 2-cell from (f, f', ϕ) to (g, g', ψ) consists of 2-cells $\xi : f \rightarrow g$ and $\xi' : f' \rightarrow g'$ equipped with an invertible 3-cell Ξ between

$$\begin{array}{ccc} A & \xrightarrow{a} & A' \\ f \downarrow \left(\begin{array}{c} \xi \\ \Rightarrow \end{array} \right) g & & \downarrow \left(\begin{array}{c} \psi \\ \Rightarrow \end{array} \right) g' \\ B & \xrightarrow{b} & B' \end{array} \quad \begin{array}{ccc} A & \xrightarrow{a} & A' \\ f \downarrow \left(\begin{array}{c} \phi \\ \Rightarrow \end{array} \right) & & \downarrow \left(\begin{array}{c} \xi' \\ \Rightarrow \end{array} \right) g' \\ B & \xrightarrow{b} & B' \end{array}$$

- a 3-cell from (ξ, ξ', Ξ) to (ζ, ζ', Z) is a pair of 3-cells $M : \xi \rightarrow \zeta$ and $M' : \xi' \rightarrow \zeta'$ satisfying the evident compatibility condition.

This is made into a **Gray**-category in the obvious way. One now checks that

- a 1-cell (f, f', ϕ) in \mathbb{PB} is a biequivalence if and only if f and f' are biequivalences in \mathcal{B}
- a 2-cell (ξ, ξ', Ξ) in \mathbb{PB} is an equivalence if and only if ξ and ξ' are equivalences in \mathcal{B}
- a 3-cell (M, M') in \mathbb{PB} is invertible if and only if M and M' are so in \mathcal{B} .

There are evident projections $P, P' : \mathbb{PB} \rightarrow \mathbb{B}$ which are **Gray**-functors, and the diagonal $\Delta : \mathbb{B} \rightarrow \mathbb{B} \times \mathbb{B}$ factorizes as

$$\mathbb{B} \xrightarrow{D} \mathbb{PB} \xrightarrow{\left(\begin{array}{c} P \\ P' \end{array} \right)} \mathbb{B} \times \mathbb{B}$$

It follows easily from the characterization of biequivalences, equivalences, and isomorphisms in \mathbb{PB} given above, that $\left(\begin{array}{c} P \\ P' \end{array} \right)$ is a fibration. It is also straightforward to check that D is a weak equivalence. \square

We now complete the proof of the theorem using a standard argument. We are to show that the trivial cofibrations are precisely the cofibrations which are weak equivalences.

Suppose first that F is a trivial cofibration; we have already observed that F is a cofibration, so we must show that it is also a weak equivalence. Since \mathbb{A} (like any object) is fibrant, there exists a map $G : \mathbb{B} \rightarrow \mathbb{A}$ with $GF = 1$.

Now F is a trivial cofibration and $\left(\begin{array}{c} P \\ P' \end{array} \right)$ a fibration, so there is a lifting H as in

$$\begin{array}{ccccc} \mathbb{A} & \xrightarrow{F} & \mathbb{B} & \xrightarrow{D} & \mathbb{PB} \\ F \downarrow & & & \nearrow H & \downarrow \left(\begin{array}{c} P \\ P' \end{array} \right) \\ \mathbb{B} & & & \xrightarrow{\left(\begin{array}{c} 1 \\ FG \end{array} \right)} & \mathbb{B} \times \mathbb{B} \end{array}$$

Since D is a weak equivalence and $PD = P'D = 1$, both P and P' are weak equivalences. Since $PH = 1$ and P is a weak equivalence, H is a weak equivalence. Since $P'H = FG$ and P' and H are weak equivalences, FG is a weak equivalence.

Finally F is a retract of FG , since $GF = 1$, and so F is indeed a weak equivalence. This proves that every trivial cofibration is a weak equivalence and a cofibration; we now turn to the converse.

If F is both a weak equivalence and a cofibration, factorize it as $F = QJ$, where J is a trivial cofibration and Q a fibration. Now F and J are weak equivalences, so Q is a weak equivalence, and hence a trivial fibration. But now since F is a cofibration, and factorizes through Q by J , it is a retract of J and hence is itself a trivial cofibration.

This completes the proof of Theorem 2.2. \square

5 Gray-groupoids

In this section we restrict from general **Gray**-categories to **Gray**-groupoids: these are **Gray**-categories in which all 1-cells, 2-cells, and 3-cells are (strictly) invertible. The **Gray**-groupoids form a full subcategory **Gray-Gpd** of **Gray-Cat**. Since the structure of **Gray**-category and that of **Gray**-groupoid are both essentially algebraic, and the inclusion is given by forgetting some of this algebraic structure, it has a left adjoint, and so the full subcategory is reflective. We shall show that the model structure on **Gray-Cat** restricts to **Gray-Gpd**.

Gray-groupoids were called *algebraic homotopy 3-types* in [4], and we shall see that they do indeed provide a model for homotopy 3-types — this is due originally to Joyal and Tierney, in unpublished but widely advertised work; here, we shall give a functorial and model-theoretic proof. The relationship between **Gray**-groupoids and homotopy 3-types has also been studied in [1] and [14]; see the introduction for comments about the relationship with [1].

Theorem 5.1 *There is a combinatorial model structure on **Gray-Gpd** for which a morphism is a fibration or weak equivalence if and only if it is one in **Gray-Cat**.*

PROOF: The cofibrations are then morphisms with the left lifting property with respect to the trivial fibrations, and the trivial cofibrations are the morphisms with the left lifting property with respect to the fibrations. The existence of the weak factorization systems is immediate. Once again every trivial cofibration is a cofibration; we have to show that a cofibration is trivial if and only if it is a weak equivalence. This will follow exactly as before provided that we can construct path objects in **Gray-Gpd**; and this will certainly be the case if our path object $\mathbb{P}\mathbb{B}$ (in **Gray-Cat**) is a **Gray**-groupoid whenever \mathbb{B} is one.

Suppose then that \mathbb{B} is a **Gray**-groupoid, and consider the **Gray**-category $\mathbb{P}\mathbb{B}$. A 3-cell (between specified 2-cells) consists of a pair (M, M') of 3-cells in \mathbb{B} satisfying a compatibility condition. Since \mathbb{B} is a **Gray**-groupoid, both M and M' are invertible, and so the 3-cell (M, M') in $\mathbb{P}\mathbb{B}$ is invertible. A 2-cell, between specified 1-cells $(f, \phi) : a \rightarrow b$ and $(f', \phi') : a \rightarrow b$, consists of 2-cells $\xi : f \rightarrow g$ and $\xi' : f' \rightarrow g'$ in \mathbb{B} equipped with an invertible 3-cell Ξ . Since \mathbb{B} is a **Gray**-groupoid, ξ and ξ' have inverses ξ^{-1} and ξ'^{-1} , and these become an inverse to (ξ, ξ', Ξ) when equipped with the 3-cell $\xi'^{-1}a.\Xi^{-1}.b\xi^{-1} : \phi.b\xi^{-1} \cong \xi'^{-1}a.\psi$. Thus all 2-cells in $\mathbb{P}\mathbb{B}$ are invertible. It remains to show that every 1-cell in $\mathbb{P}\mathbb{B}$ is invertible. A 1-cell from $a : A \rightarrow A'$ to $b : B \rightarrow B'$ consists of 1-cells $f : A \rightarrow B$ and $f' : A' \rightarrow B'$ in \mathbb{B} , equipped with an equivalence $\phi : bf \simeq f'a$. But since \mathbb{B} is a **Gray**-groupoid, f and f' have inverses f^{-1} and f'^{-1} , and ϕ is not just an equivalence but an isomorphism; and now f^{-1} and f'^{-1} become an inverse for (f, f', ϕ) when equipped with the 2-cell $f'^{-1}.\phi^{-1}.f^{-1} : af^{-1} = f'^{-1}f'af^{-1} \cong f'^{-1}bf^{-1} = f'^{-1}b$. \square

In this restricted version, the description of fibrations becomes simpler. A **Gray**-functor $P : \mathbb{E} \rightarrow \mathbb{B}$ between **Gray**-groupoids is a fibration when the following conditions hold:

- for every $E \in \mathbb{E}$ and every 1-cell $f : B \rightarrow PE$ in \mathbb{B} there is a lifting $f' : D \rightarrow E$ in \mathbb{E}
- for every $f : D \rightarrow E$ in \mathbb{E} and every 2-cell $\beta : b \rightarrow Pf$ in \mathbb{B} there is a lifting $\gamma : d \rightarrow f$ in \mathbb{E}
- for every 2-cell $\alpha : f \rightarrow g : D \rightarrow E$ in \mathbb{E} and every 3-cell $M : \beta \rightarrow P\alpha$ in \mathbb{B} there is a lifting $M' : \gamma \rightarrow \alpha$ in \mathbb{E}

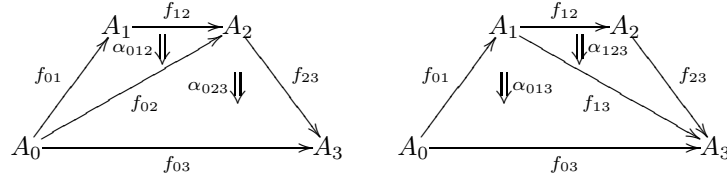
A **Gray**-functor $F : \mathbb{E} \rightarrow \mathbb{B}$ between **Gray**-groupoids is a weak equivalence when the following conditions hold:

- for every $B \in \mathbb{B}$ there is an $E \in \mathbb{E}$ and a morphism $f : B \rightarrow FE$
- for all $D, E \in \mathbb{E}$ and every $b : FD \rightarrow FE$ in \mathbb{B} there is a morphism $f : D \rightarrow E$ in \mathbb{E} and a 2-cell $\beta : b \rightarrow Pf$ in \mathbb{B}
- for all $\alpha, \beta : f \rightarrow g : D \rightarrow E$ in \mathbb{E} and every 3-cell $Y : P\alpha \rightarrow P\beta$ in \mathbb{B} there is a 3-cell $X : \alpha \rightarrow \beta$ with $PX = Y$.

In [1], an adjunction between Gray-groupoids and simplicial sets was described, with the right adjoint $N : \mathbf{Gray-Gpd} \rightarrow \mathbf{SSet}$ giving the nerve of a **Gray**-groupoid; we shall write Π_3 for the left adjoint.

Proposition 5.2 *The nerve functor $N : \mathbf{Gray-Gpd} \rightarrow \mathbf{SSet}$ preserves fibrations and trivial fibrations, and so is the right adjoint part of a Quillen adjunction.*

PROOF: Suppose first that $F : \mathbb{A} \rightarrow \mathbb{B}$ is a trivial fibration in **Gray-Gpd**. We must show that $NF : N\mathbb{A} \rightarrow N\mathbb{B}$ is a trivial fibration of simplicial sets; in other words, that it has the right lifting property with respect to the inclusion $\partial\Delta[n] \rightarrow \Delta[n]$ for all n . For $n = 0$ this is the fact that F is surjective on objects and for $n = 1$ it is the fact that F is full on 1-cells. For $n = 2$ it says that for any 1-cells $f : A \rightarrow B$, $g : B \rightarrow C$ and $h : A \rightarrow C$ in \mathbb{A} , and any 2-cell $\beta : Fg.Ff \rightarrow Fh$ in \mathbb{B} , there is a 2-cell $\alpha : gf \rightarrow h$ in \mathbb{A} with $F\alpha = \beta$; this holds since F is full on 2-cells (F is locally full). For $n = 3$ it becomes a little more complicated. Consider 2-cells in \mathbb{A} as in the diagram



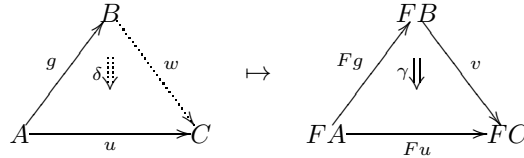
where the left composite may be written as $\alpha_{023}.f_{23}\alpha_{012}$ and the right composite as $\alpha_{013}.\alpha_{123}f_{01}$. Apply F to each side, and suppose that we have a 3-cell $Y : F(\alpha_{023}.f_{23}\alpha_{012}) \rightarrow F(\alpha_{013}.\alpha_{123}f_{01})$ between the resulting 2-cells in \mathbb{B} . The case $n = 3$ amounts to the fact that there is a 3-cell $X : \alpha_{023}.f_{23}\alpha_{012} \rightarrow \alpha_{013}.\alpha_{123}f_{01}$ in \mathbb{A} with $FX = Y$; this is true since $F : \mathbb{A} \rightarrow \mathbb{B}$ is full on 3-cells (locally locally full). For $n > 3$, an n -simplex in the nerve of a **Gray**-groupoid does not involve further data, it just amounts to the assertion that two pasting composites (with the same 2-cells as domain and codomain) of 3-cells are equal. Since F is faithful on 3-cells (locally locally faithful), such assertions can be lifted from \mathbb{B} to \mathbb{A} , and so all remaining conditions hold. This proves that NF is a trivial fibration.

Suppose now that $F : \mathbb{A} \rightarrow \mathbb{B}$ is a fibration in **Gray-Gpd**. We must show that $NF : N\mathbb{A} \rightarrow N\mathbb{B}$ is a fibration of simplicial sets (a Kan fibration); in other words, that it has the left lifting property with respect to the horns $\Lambda^r[n] \rightarrow \Delta[n]$.

For $n = 1$, this says that given $A \in \mathbb{A}$ and a 1-cell $f : B \rightarrow FA$ in \mathbb{B} there is a 1-cell $f' : A' \rightarrow \mathbb{A}$ with $FA' = B$ and $Ff' = f$, as well as a corresponding statement involving 1-cells $FA \rightarrow B$. This first is part of our characterization of fibrations; the second an easy consequence of it, given that every 1-cell is invertible.

For $n = 2$ and $r = 1$, this says that given $f : A \rightarrow B$ and $g : B \rightarrow C$ in \mathbb{A} and a 2-cell $\beta : Fg.Ff \rightarrow k$ in \mathbb{B} there is a 1-cell $h : A \rightarrow C$ with $Fh = k$ and a 2-cell $\alpha : gf \rightarrow h$ with $F\alpha = \beta$. This is once again part of our characterization of fibrations in **Gray-Gpd**.

For $n = 2$ and $r = 0$ we start with 1-cells $g : A \rightarrow B$ and $u : A \rightarrow C$ in \mathbb{A} , and a 1-cell $v : FB \rightarrow FC$ and 2-cell $\gamma : v.Ff \rightarrow Fu$ in \mathbb{B} , as in the solid parts of the diagram below.

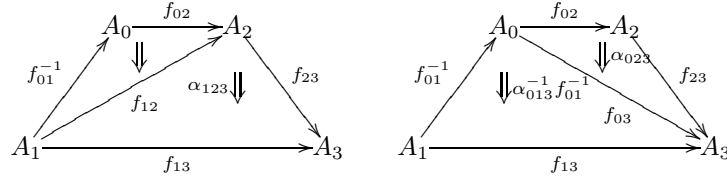


We must construct a 1-cell $w : B \rightarrow C$ in \mathbb{A} with $Fw = v$ and a 2-cell $\delta : wg \rightarrow u$ with $F\delta = \gamma$. Let $f = u^{-1}$ and $k = v^{-1}$. Then $v^1.v.Fg.Ff = Fg.Ff$ and $v^{-1}.Fu.Ff = k.Fu.Fu^{-1} = k$, and so the 2-cell $v^{-1}.\gamma.Ff$ goes from $Fg.Ff$ to k . By the previous paragraph there is a 2-cell $\alpha : gf \rightarrow h$ in \mathbb{A} with $Fh = k$ and $F\alpha = v^{-1}.\gamma.Ff$. Let $w = h^{-1}$ and $\delta = w\alpha u : wgu^{-1}u = wg$ and $whu = h^{-1}hu = u$, and so δ is indeed a 2-cell from wg to u . Also $Fw = Fh^{-1} = (Fh)^{-1} = k^{-1} = v$, while $F\delta = Fw.F\alpha.Fu = Fw.v^{-1}.\gamma.Ff.Fu = Fh^{-1}.k.\gamma.Fu^{-1}.Fu = \gamma$, and so the condition does hold. The case $n = r = 2$ is similar.

For $n = 3$, we start with objects A_0, A_1, A_2 , and A_3 of \mathbb{A} , with 1-cells $f_{ij} : A_i \rightarrow A_j$ for $0 \leq i < j \leq 3$. For $0 \leq i < j < k \leq 3$ with $r \in \{i, j, k\}$ we have a 2-cell $\alpha_{ijk} : f_{jk}f_{ij} \rightarrow f_{ik}$ in \mathbb{A} . For the choice of $0 \leq i < j < k \leq 3$ with $r \notin \{i, j, k\}$ there is a 2-cell $\beta_{ijk} : F(f_{jk}f_{ij}) \rightarrow Ff_{ik}$ in \mathbb{B} , and finally a 3-cell Y between the two pasting composites.

We treat the case $r = 1$ in detail. The “missing 2-cell” is $\alpha_{023} : f_{23}f_{02} \rightarrow f_{03}$. Since α_{012} is an equivalence, we can find a 2-cell $\alpha'_{023} : f_{23}f_{02} \rightarrow f_{03}$ and a 3-cell $X' : \alpha'_{023} \cdot f_{23}\alpha_{012} \rightarrow \alpha_{013} \cdot \alpha_{123}f_{01}$. Applying F gives a 3-cell $FX' : F\alpha'_{023} \cdot F(f_{23}\alpha_{012}) \rightarrow F\alpha_{013} \cdot F(\alpha_{123}f_{01})$, but we also have a 3-cell $Y : \beta_{023} \cdot F(f_{23}\alpha_{012}) \rightarrow F\alpha_{013} \cdot F(\alpha_{123}f_{01})$, and now there is a unique 3-cell $Z : \beta_{023} \rightarrow F\alpha'_{023}$ which pastes onto FX' to give Y . We can lift this to a 3-cell $W : \alpha_{023} \rightarrow \alpha'_{023}$ with $F\alpha_{023} = \beta_{023}$ and $FW = Z$. Finally pasting W onto X' gives a 3-cell $X : \alpha_{023} \cdot f_{23}\alpha_{012} \rightarrow \alpha_{013} \cdot \alpha_{123}f_{01}$ over Y .

The case $r = 2$ is similar to that of $r = 1$. For $r = 0$, where α_{123} is the missing face, we can take inverses and work instead with



where the unnamed face on the left is $\alpha_{012}^{-1}f_{01}^{-1}$; which reduces it to the case $r = 1$. Similarly the case $r = 3$ can be reduced to $r = 2$ by taking inverses.

Since we are dealing with nerves of **Gray**-groupoids, all horns have unique fillers for $n \geq 4$, and so the condition is automatic. \square

By Ken Brown’s lemma [8, Lemma 1.1.12], N takes all weak equivalences between fibrant objects to weak equivalences; but all objects of **Gray-Gpd** are fibrant, and so in fact N also preserves weak equivalences. By a similar argument, using the fact that all objects of **SSet** are cofibrant, Π_3 also preserves weak equivalences. We record this as

Proposition 5.3 *Both the nerve functor $N : \mathbf{Gray-Gpd} \rightarrow \mathbf{SSet}$ and its left adjoint $\Pi_3 : \mathbf{SSet} \rightarrow \mathbf{Gray-Gpd}$ preserve weak equivalences.* \square

Quillen adjunctions induce derived adjunctions between the homotopy categories. Since Π_3 preserves weak equivalences and all objects of **SSet** are cofibrant, the simplicial sets X for which the unit of the derived adjunction is invertible are precisely those for which the unit $X \rightarrow N\Pi_3X$ of the Quillen adjunction is a weak equivalence. Berger showed in [1, Proposition 3.2] that for a simplicial set X , the unit $X \rightarrow N\Pi_3X$ is a weak equivalence if and only if X is a 3-type.

Similarly, since N preserves weak equivalences and all objects of **Gray-Gpd** are fibrant, the **Gray**-groupoids \mathbb{A} for which the counit of the derived adjunction is invertible are precisely those for which the counit $\Pi_3N\mathbb{A} \rightarrow \mathbb{A}$ is a weak equivalence. Once we have shown that this holds for all **Gray**-groupoids, then we shall have constructed an equivalence between the homotopy categories of **Gray**-groupoids and the homotopy category of simplicial 3-types.

Theorem 5.4 *The Quillen adjunction between **Gray-Gpd** and **SSet** induces an equivalence between the homotopy categories of algebraic homotopy 3-types and simplicial homotopy 3-types.*

PROOF: We need to show that the counit map $E : \Pi_3N\mathbb{A} \rightarrow \mathbb{A}$ is a triequivalence of **Gray**-categories. An object of $\Pi_3N\mathbb{A}$ is a 0-simplex of $N\mathbb{A}$; that is, an object of \mathbb{A} . Thus $\Pi_3N\mathbb{A}$ and \mathbb{A} have the same objects, and the counit map $E : \Pi_3N\mathbb{A} \rightarrow \mathbb{A}$ is the identity on objects.

The 1-cells of $\Pi_3N\mathbb{A}$ are freely generated by the non-degenerate 1-simplices of $N\mathbb{A}$ together with formal inverses; that is, by the non-identity 1-cells of \mathbb{A} together with formal inverses. We write a typical such 1-cell as \mathbf{f} or $[f_n] \dots [f_1]$, where the f_i are 1-cells in \mathbb{A} or their formal inverses. Any non-identity 1-cell in \mathbb{A} can

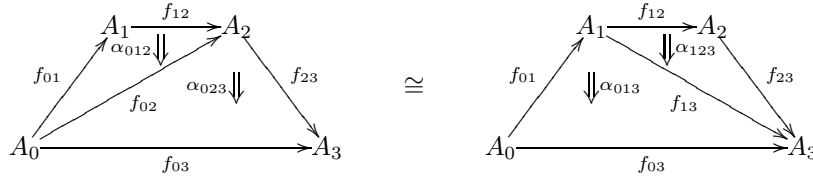
be realized as the composite of such a word of length 1 (or 0 in the case of an identity), and so the counit $E : \Pi_3 N\mathbb{A} \rightarrow \mathbb{A}$ is surjective (and full) on 1-cells.

The 2-cells of $\Pi_3 N\mathbb{A}$ are obtained by freely adjoining an invertible 2-cell $[f_2][f_1] \rightarrow [f]$ for each 2-cell $\phi : f_2 f_1 \rightarrow f$ which is “non-degenerate” in the sense that no more than one of f_1 , f_2 , and ϕ are identities.

Now a general 1-cell in $\Pi_3 N\mathbb{A}$ is a word $[f_n] \dots [f_2][f_1]$ in the 1-cells of \mathbb{A} and their formal inverses. But in fact any such 1-cell is isomorphic in $\Pi_3 N\mathbb{A}$ to an identity or to a word consisting of a single 1-cell in \mathbb{A} . To see this, first observe that if $f : A \rightarrow B$ is any 1-cell in \mathbb{A} , then it has an inverse, say g , and now the equations $gf = 1$ and $fg = 1$ can be seen 2-simplices, and so force g to be isomorphic to the formal inverse f^{-1} . So any word $[f_n] \dots [f_2][f_1]$ is isomorphic to one not involving any formal inverses. But now the word $[f_2][f_1]$ is isomorphic to the actual composite $[f_2 f_1]$, and by an easy induction any longer word is likewise isomorphic to an arrow in \mathbb{A} .

We now turn to the fullness on 2-cells of $E : \Pi_3 N\mathbb{A} \rightarrow \mathbb{A}$. It suffices to show fullness on 2-cells $[f] \rightarrow [g]$, where f and g are 1-cells in \mathbb{A} . Any 2-cell $\phi : [f] \rightarrow [g]$ in \mathbb{A} can be regarded as a 2-cell $\phi : [f][1] = [f] \rightarrow [g]$, and so we do indeed have fullness on 2-cells.

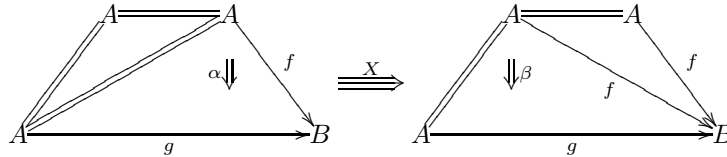
The 3-cells of $\Pi_3 N\mathbb{A}$ are once again obtained by adjoining invertible 3-cells



between the formal composites on either side of the diagram, for each 3-cell in \mathbb{A} between the actual composites. But this time it is not done freely; rather we introduce an equation for each (commutative) 4-simplex in $N\mathbb{A}$. (We have not mentioned here the fact that there will generally already be some 3-cells — the pseudonaturality isomorphisms whose presence is forced whenever there are 2-cells whose horizontal composite is not fully determined.)

α
Consider a pair of 2-cells $\alpha, \beta : f \rightarrow g$ in $\Pi_3 N\mathbb{A}$. We wish to show that $E : \Pi_3 N\mathbb{A} \rightarrow \mathbb{A}$ is fully faithful on 3-cells from α to β . We have already seen that f and g can be replaced by isomorphic 1-cells $[f]$ and $[g]$ coming from \mathbb{A} . The fully faithfulness on 3-cells will not be affected by whiskering by the isomorphisms $[f] \cong f$ and $[g] \cong g$, so we may as well suppose that $\alpha, \beta : [f] \rightarrow [g]$. Furthermore, we can inductively build up isomorphisms $[\alpha] \cong \alpha$ and $[\beta] \cong \beta$ in $\Pi_3 N\mathbb{A}$, where $\alpha, \beta : f \rightarrow g$ are now 2-cells in \mathbb{A} . Once again, it will suffice to prove fully faithfulness on 3-cells $[\alpha] \rightarrow [\beta]$.

Any 3-cell $X : \alpha \rightarrow \beta$ in \mathbb{A} can be seen as a (fairly degenerate) 3-simplex



and so can be realized via E . This gives fullness on 3-cells of E .

Suppose that $X : [\alpha] \rightarrow [\beta]$ is any 3-cell in $\Pi_3 N\mathbb{A}$. Then X can be built up as a formal composite of 3-cells in \mathbb{A} ; this involves whiskering by 2-cells or 1-cells, and vertical composition of 3-cells. We can always use the previous arguments to restrict to the case where these 3-cells have domain and codomain 2-cells coming from \mathbb{A} (in other words, of the form $[\gamma]$ or $[\delta]$ for 2-cells γ or δ in \mathbb{A}).

If X itself has the form $[Y]$ for some 3-cell $Y : \alpha \rightarrow \beta$ in \mathbb{A} , then clearly $X = EX = E[Y] = Y$, and so $X = [Y] = [EX]$. We now prove inductively that $X = [EX]$ for any $X : [\alpha] \rightarrow [\beta]$. Any such X has the form $[X_n] \dots [X_1]$; for simplicity we give only the case $X = [X_2][X_1]$; the general inductive step is essentially

There is a 4-simplex

than a simplicial structure. We can build a simplicial model “ $\mathbf{2}_2$ ” of a parallel pair



by taking the coproduct $\Delta_1 + \Delta_1$ and identifying the boundaries. Similarly we can build a simplicial model “ $\mathbf{2}_2$ ” of a (globular) 2-cell



by collapsing one of the faces of a 2-simplex. The inclusion $\mathbf{2}_2 \rightarrow \mathbf{2}_2$ is a monomorphism, so its image under Π_3 is a cofibration; it follows that F is full on 2-cells.

Fullness on 3-cells is once again proved by constructing a simplicial model of a parallel pair of 2-cells, and a simplicial model of a 3-cell, and observing that the inclusion is a monomorphism.

Finally, faithfulness on 3-cells can be interpreted as “fullness on 4-cells”: think of a Gray-category as some sort of 4-dimensional category in which there are no non-identity 4-cells; then equality of 3-cells becomes the existence of a (necessarily trivial) 4-cell between them. Let $f, g : A \rightarrow B$ be 1-cells in \mathbb{A} , with 2-cells $\alpha, \beta : f \rightarrow g$ and 3-cells $M, M' : \alpha \rightarrow \beta$. We need to show that $FM = FM'$ implies that $M = M'$; in other words that any 4-cell $FM \rightarrow FM'$ is in the image under F of a 4-cell $M \rightarrow M'$. This can once again be expressed simplicially. \square

6 Localizations of \mathbf{SSet}

In this section we describe the relationship between n -groupoids and homotopy n -types (for small n) in terms of Bousfield localizations [7] of \mathbf{SSet} . For each n , let g_n be the inclusion $\partial\Delta_{n+2} \rightarrow \Delta_{n+2}$. The localizations we consider will be with respect to these maps g_n . (Once could also use the image of the inclusion $S^{n+1} \rightarrow D^{n+2}$ in \mathbf{Top} under the singular functor $\mathbf{Top} \rightarrow \mathbf{SSet}$ as g_n .)

We start by recalling the well-understood situation for $n = 1$. The nerve functor $N : \mathbf{Gpd} \rightarrow \mathbf{SSet}$ for groupoids has a left adjoint $\Pi_1 : \mathbf{SSet} \rightarrow \mathbf{Gpd}$ sending a simplicial set to its fundamental groupoid [2]. The nerve functor preserves fibrations and trivial fibrations, so is a right Quillen functor. The left adjoint Π_1 therefore preserves all cofibrations and trivial cofibrations. But since all objects of \mathbf{SSet} are cofibrant, and all objects of \mathbf{Gpd} are fibrant, in fact both functors preserve weak equivalences. The fundamental groupoid $\Pi_1(X)$ of a simplicial set agrees with the ordinary fundamental groupoid $\Pi_1|X|$ of the geometric realization of X .

Now the nerve functor is fully faithful, so the counit $\Pi_1 N \rightarrow 1$ is invertible, and so Π_1 can be thought of as a reflection onto a full subcategory. The unit is not invertible, and Π_1 does destroy information: in particular, it destroys all homotopical information in dimension greater than 1. More precisely, for a morphism $f : X \rightarrow Y$ of simplicial sets, the induced functor $\Pi_1(f) : \Pi_1(X) \rightarrow \Pi_1(Y)$ is an equivalence if and only if f induces a bijection $\pi_0 f : \pi_0 X \rightarrow \pi_0 Y$ between the sets of connected components, and an isomorphism $\pi_1 f : \pi_1(X, x) \rightarrow \pi_1(Y, fx)$ of fundamental groups for all choices of basepoint $x \in X$.

Recall that g_1 is the inclusion $\partial\Delta_3 \rightarrow \Delta_3$. Let $P_1\mathbf{SSet}$ be the Bousfield localization of \mathbf{SSet} with respect to the map g_1 . The weak equivalences are precisely the morphisms $f : X \rightarrow Y$ of simplicial sets for which $\Pi_1(f) : \Pi_1(X) \rightarrow \Pi_1(Y)$ is an equivalence, and the cofibrations are the usual cofibrations of simplicial sets (the monomorphisms). Thus the Quillen adjunction between \mathbf{Gpd} and \mathbf{SSet} passes to a Quillen adjunction between \mathbf{Gpd} and $P_1\mathbf{SSet}$, which is now a Quillen equivalence.

Next we turn to the case $n = 2$. Once again there is a nerve functor $N : \mathbf{2-Gpd} \rightarrow \mathbf{SSet}$. This nerve of a 2-groupoid is a special case of Street’s nerve of a bicategory. As observed in [17], this nerve functor $N : \mathbf{2-Gpd} \rightarrow \mathbf{SSet}$ is the right adjoint part of a Quillen adjunction. We shall write Π_2 for the left adjoint, called W in [17]. Once again N and Π_2 both preserve weak equivalences; a simplicial map $f : X \rightarrow Y$ induces a weak equivalence of 2-groupoids if and only if it induces a bijection $\pi_0(f) : \pi_0(X) \rightarrow \pi_0(Y)$ of sets,

and isomorphisms $\pi_1(f) : \pi_1(X, x) \rightarrow \pi_1(Y, fx)$ and $\pi_2(f) : \pi_2(X, x) \rightarrow \pi_2(Y, y)$ of groups for all basepoints $x \in X$.

Let $P_2\mathbf{SSet}$ be the Bousfield localization of \mathbf{SSet} with respect to the map g_2 . This has the same cofibrations as \mathbf{SSet} , and $f : X \rightarrow Y$ is a weak equivalence in $P_2\mathbf{SSet}$ if and only if $\Pi_2(f) : \Pi_2(X) \rightarrow \Pi_2(Y)$ is one in $\mathbf{2-Gpd}$. The Quillen adjunction $\Pi_2 \dashv N : \mathbf{2-Gpd} \rightarrow \mathbf{SSet}$ passes to a Quillen equivalence between $\mathbf{2-Gpd}$ and $P_2\mathbf{SSet}$.

Finally, we turn to the case $n = 3$. We saw in the previous section that the nerve functor $N : \mathbf{Gray-Gpd} \rightarrow \mathbf{SSet}$ of [1] is the right adjoint part of a Quillen adjunction; we write Π_3 for the left adjoint part. Once again N and Π_3 both preserve weak equivalences; a simplicial map $f : X \rightarrow Y$ induces a weak equivalence of Gray-groupoids if and only if it induces a bijection $\pi_0(f) : \pi_0(X) \rightarrow \pi_0(Y)$ of sets, and isomorphisms $\pi_n(f) : \pi_n(X, x) \rightarrow \pi_n(Y, fx)$ of groups for all basepoints $x \in X$, and all $n \in \{1, 2, 3\}$.

Let $P_3\mathbf{SSet}$ be the Bousfield localization of \mathbf{SSet} with respect to the map g_3 . This has the same cofibrations as \mathbf{SSet} , and $f : X \rightarrow Y$ is a weak equivalence in $P_3\mathbf{SSet}$ if and only if $\Pi_3(f) : \Pi_3(X) \rightarrow \Pi_3(Y)$ is one in $\mathbf{Gray-Gpd}$. The Quillen adjunction $\Pi_3 \dashv N : \mathbf{Gray-Gpd} \rightarrow \mathbf{SSet}$ passes to a Quillen equivalence between $\mathbf{Gray-Gpd}$ and $P_3\mathbf{SSet}$.

7 Tricategories

We conjecture that the model structure on $\mathbf{Gray-Cat}$ can be extended to the context of tricategories. To do this, one should use a “fully algebraic” definition of tricategories, such as that of Gurski [6], so that the category \mathbf{Tricat} of tricategories and strict morphisms of tricategories is locally presentable, and contains $\mathbf{Gray-Cat}$ as a full reflective subcategory.

The definitions of weak equivalence and fibration still make perfectly good sense in the context of tricategories; indeed the notion of triequivalence (our weak equivalences) was first defined in that context [4]. Choosing these weak equivalences and fibrations determines a model structure uniquely if one exists. We conjecture that it does. The inclusion of $\mathbf{Gray-Cat}$ into \mathbf{Tricat} preserves fibrations and weak equivalences, and so would be the right adjoint part of a Quillen adjunction.

We conjecture further that this Quillen adjunction is in fact a Quillen equivalence. Since the inclusion $I : \mathbf{Gray-Cat} \rightarrow \mathbf{Tricat}$ is fully faithful, the counit $LI \rightarrow 1$ of the adjunction is invertible. Since all objects of $\mathbf{Gray-Cat}$ are fibrant, the unit of the derived adjunction will also be invertible; so it would remain only to show that the counit of the derived adjunction is invertible. This will be the case provided that the counit $\mathbb{T} \rightarrow IL\mathbb{T}$ is a weak equivalence for every cofibrant tricategory \mathbb{T} , and Michael Makkai has announced [16] that he has proved this to be the case.

On the other hand, we could consider strict 3-categories. The category $\mathbf{3-Cat}$ of 3-categories and 3-functors is a full reflective subcategory of $\mathbf{Gray-Cat}$. The model structure on $\mathbf{Gray-Cat}$ lifts across the adjunction to give a model structure on $\mathbf{3-Cat}$ for which a 3-functor is a weak equivalence or a fibration if and only if the corresponding \mathbf{Gray} -functor is one. One simply applies the reflection into $\mathbf{Gray-Cat}$ to each of the generating cofibrations and generating trivial cofibrations (in fact this is only needed for one generating trivial cofibration; the other generating trivial cofibrations and the generating cofibrations are already in $\mathbf{3-Cat}$) and then observes that for a 3-category \mathbb{B} , the path object \mathbb{PB} in $\mathbf{Gray-Cat}$ constructed above is in fact a 3-category.

The inclusion of $\mathbf{3-Cat}$ in $\mathbf{Gray-Cat}$ together with its left adjoint are then a Quillen adjunction, by construction, but they will not be a Quillen equivalence, essentially because not every 3-category is triequivalent to a 3-category [4].

8 Computads and sesquicategories

We have given explicit descriptions of the fibrations and weak equivalences, but so far the cofibrations have been defined only via a left lifting property. In the following section we investigate the cofibrations of

Gray-Cat, and in particular, the cofibrant objects. First, however, we develop some necessary material on sesquicategories [18].

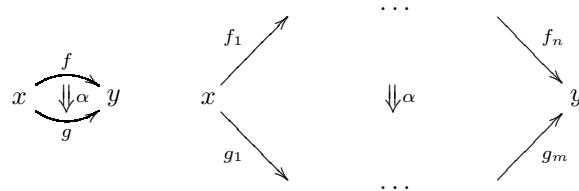
First, recall that the functor $U : \mathbf{2-Cat} \rightarrow \mathbf{Cat}$ which forgets the 2-cells of a 2-category has both a left adjoint D , which adds identity 2-cells to a category, and a right adjoint C , which adds a single (invertible) 2-cell between each parallel pair of 1-cells. Both D and C are fully faithful, so the unit $1 \rightarrow UD$ of the adjunction $D \dashv U$ and the counit $UC \rightarrow 1$ of the adjunction $U \dashv C$ are invertible.

We shall regard the category **2-Cat** as monoidal via the Gray tensor product, and **Cat** as monoidal via the “funny” tensor product \square . Recall, for example from [18], that \square is part of a monoidal closed structure in which the internal hom $[A, B]$ has as objects the functors from A to B and as morphisms from f to g , the set of all $\text{ob}A$ -indexed families of morphisms $fa \rightarrow ga$, with no naturality condition imposed.

This determines the funny tensor product; it can be described more explicitly as follows: its objects are pairs (a, b) of objects of A and B , while its morphisms are freely generated by morphisms of the form $(\alpha, 1_b)$ and $(1_a, \beta)$, where $\alpha : a \rightarrow a'$ is a morphism in A and $\beta : b \rightarrow b'$ a morphism in B ; subject to relations which state that $(a, -)$ and $(-, b)$ are functors. Another way to say this is $A \square B = U(CA \otimes CB) = U(DA \otimes DB)$; and in fact both U and C are strong monoidal functors, while D is only opmonoidal.

Now a category enriched in **2-Cat** (with respect to the Gray tensor product) is of course a Gray-category, while a category enriched in **Cat** (with respect to the funny tensor product) is called a *sesquicategory*. Since the adjunction $U \dashv C$ is monoidal, we obtain an adjunction $U_* \dashv C_*$ between **Gray-Cat** and **SesquiCat**, where U_* discards the 3-cells, and C_* adjoins a single (invertible) 3-cell between each parallel pair of 2-cells. Since D is not monoidal it does not induce a 2-functor D_* , but there is nonetheless a left adjoint L to U_* , which adjoins only identity 2-cells and pseudonaturality isomorphisms. Note that the counits $LU_*\mathcal{A} \rightarrow \mathcal{A}$ of the adjunction $L \dashv U_*$ are all bijective on objects, 1-cells, and 2-cells; they only change the 3-cells.

We shall also need a notion of *free* sesquicategory on a computad. Recall, for example from [18], that a computad consists of following structure: first of all a directed graph $d, c : G_1 \rightarrow G_0$, on which we form the free category C ; then a second directed graph $d, c : G_2 \rightarrow C_1$, whose vertices are the 1-cells of C ; finally we require that the “globular relations” $dd = dc$ and $cd = cc$ hold, so that elements of G_2 (called 2-cells) can be written as the diagram below on the left where $f, g : x \rightarrow y$ are 1-cells in C or in expanded from as in the diagram on the right



in which the ellipses denote an unspecified number of further composites (the picture on the right appears to suggest that $n \geq 2$, but in fact n could be 0 or 1). A morphism from such a computad G to a computad H consists of a graph morphism f from $d, c : G_1 \rightarrow G_0$ to $d, c : H_1 \rightarrow H_0$, consisting of $f_0 : G_0 \rightarrow H_0$ and $f_1 : G_1 \rightarrow H_1$ compatible with the source and target maps; equipped with a map $f_2 : G_2 \rightarrow H_2$ satisfying the evident compatibility condition.

These form a category **Cmptd** which, as observed by Schanuel, is in fact a presheaf category. There is a forgetful functor $V : \mathbf{SesquiCat} \rightarrow \mathbf{Cmptd}$ which sends a sesquicategory \mathcal{A} to the computad whose 0-cells and 1-cells are the 0-cells and 1-cells of \mathcal{A} , and whose 2-cells between the paths $f_n \dots f_2 f_1 \rightarrow g_m \dots g_2 g_1$ are the 2-cells between their composites in \mathcal{A} .

This forgetful functor V has a left adjoint $H \dashv V$, which can be constructed as follows. An object of HG is an element of G_0 . A 1-cell of HG is a path in the 1-cells of G . A “basic 2-cell”, is an expression of the form $\ell \alpha r$, where $\ell : w \rightarrow x$ and $r : y \rightarrow z$ are 1-cells in HG , and $\alpha : f \rightarrow g : x \rightarrow y$ is an element of G_2 . Such a basic 2-cell has a source 1-cell $\ell f r$ and target 1-cell $\ell g r$. A 2-cell of HG is a path in the basic 2-cells.

We shall call a sesquicategory *free*, if it is (isomorphic to) one of the form HG for a computad G . In [12, Lemma 4.7] it was proved that any retract of a free category is free. We use this fact to prove an analogue for sesquicategories.

Proposition 8.1 *A retract of a free sesquicategory is free.*

PROOF: Let G be a computad, HG the free sesquicategory on G , and let $I : \mathcal{A} \rightarrow HG$ be a sesquifunctor with a retract $R : HG \rightarrow \mathcal{A}$. Certainly the underlying category of HG is free, and the underlying category of \mathcal{A} is a retract of it, so the underlying category of \mathcal{A} is also free.

As well as the underlying category of a sesquicategory, there is also the category obtained by discarding the objects: it has arrows of the sesquicategory as objects, 2-cells of the sesquicategory as arrows, and vertical composition in the sesquicategory as composition. This clearly induces a functor $W : \mathbf{SesquiCat} \rightarrow \mathbf{Cat}$. If \mathcal{A} is a retract of HG then $W\mathcal{A}$ is a retract of WHG ; but WHG is the free category on all whiskerings of 2-cells in HG , so $W\mathcal{A}$ is also free.

It remains to show that the generating arrows of $W\mathcal{A}$ are freely generated under whiskering in \mathcal{A} by some subset. Say that a 2-cell $\alpha : f \rightarrow g$ in \mathcal{A} is indecomposable if it is indecomposable as an arrow of $W\mathcal{A}$: these indecomposable 2-cells are the generating arrows of $W\mathcal{A}$. Say that such an indecomposable 2-cell is *totally indecomposable* if moreover it cannot be written as a whiskering $\alpha = g\beta f$ unless f and g are both identities. We need to show that every indecomposable 2-cell can be written uniquely as a whiskering of a totally indecomposable 2-cell.

Let $\alpha : f \rightarrow g$ be any 2-cell in \mathcal{A} , and consider its image $I\alpha : If \rightarrow Ig$ in HG . We can write $I\alpha$ uniquely as a composite $\beta_n \dots \beta_1$ of indecomposables; and each of these can in turn be written uniquely as a whiskering $\beta_i = r_i \gamma_i \ell_i$ with γ_i totally indecomposable. Since the underlying category of HG is free, each r_i and each ℓ_i has a well-defined length; write $\pi(\alpha)$ for the sum of all these lengths.

If there is an indecomposable 2-cell $\alpha' : f' \rightarrow g' : x' \rightarrow y'$ which is not the whiskering of any totally indecomposable 2-cell, then there is one with $\pi(\alpha')$ minimal. Then certainly α' is not totally indecomposable, so we can write it as a whiskering $\alpha' = r\alpha\ell$, where $\alpha : f \rightarrow g : x \rightarrow y$, and r and ℓ are not both identities. Decompose $I\alpha$ as above: $I\alpha = \beta_n \dots \beta_1$ where $\beta_i = r_i \gamma_i \ell_i$. But now the decomposition of $I\alpha'$ is $(r\beta_n\ell) \dots (r\beta_1\ell)$, with $r\beta_i\ell = rr_i\beta_i\ell_i\ell$; and so $\pi(\alpha) < \pi(\alpha')$ contradicting the minimality of $\pi(\alpha')$.

This proves that every indecomposable 2-cell is the whiskering of some totally indecomposable 2-cell. It remains to show the uniqueness. Suppose then that $\alpha : f \rightarrow g : x \rightarrow y$ and $\alpha' : f' \rightarrow g' : x' \rightarrow y'$ are totally indecomposable 2-cells in \mathcal{A} , and that $r\alpha\ell = r'\alpha'\ell'$ for some 1-cells $r : y \rightarrow z$, $r' : y' \rightarrow z$, $\ell : w \rightarrow x$, and $\ell' : w \rightarrow x'$.

Write $I\alpha = \beta_n \dots \beta_1$ and $\beta_i = r_i \gamma_i \ell_i$ where each γ_i is totally indecomposable in HG ; and similarly $I\alpha' = \beta'_n \dots \beta'_1$ and $\beta'_i = r'_i \gamma'_i \ell'_i$. Now $Ir.I\alpha.I\ell = I(r)r_n\gamma_n\ell_n I(\ell) \dots I(r)r_1\gamma_1\ell_1 I(\ell)$ and $Ir'.I\alpha'.I\ell' = I(r')r'_n\gamma'_n\ell'_n I(\ell') \dots I(r')r'_1\gamma'_1\ell'_1 I(\ell')$ are both decompositions into totally indecomposables of the same 2-cell in HG , and so must agree. Thus $n = n'$ and $\gamma_i = \gamma'_i$ for each i , while also $I(r)r_i = I(r')r'_i$ and $\ell_i I(\ell) = \ell'_i I(\ell')$ for each i .

Since $I(r)r_i = I(r')r'_i$, either there is a 1-cell s such that $I(r) = I(r')s$ and $sr_i = r'_i$, or there is a 1-cell s such that $I(r') = I(r)s$ and $sr'_i = r_i$. Without loss of generality we take the former. Applying R we get $r' = rR(s)$, and now

$$\begin{aligned} \alpha' &= RI\alpha' \\ &= R(\beta'_n) \dots R(\beta'_1) \\ &= R(r'_n\gamma_n\ell'_n) \dots R(r'_1\gamma_1\ell'_1) \\ &= R(sr_n\gamma_n\ell'_n) \dots R(sr_1\gamma_1\ell'_1) \\ &= Rs.(R(r_n\gamma_n\ell'_n) \dots R(r_1\gamma_1\ell'_1)) \end{aligned}$$

but α' was assumed totally indecomposable, and so Rs must be an identity, $r = r'$.

A similar argument shows that $\ell = \ell'$. Now

$$Ir.I\alpha.I\ell = Ir'.I\alpha'.I\ell'$$

in HG , with $r = r'$ and $\ell = \ell'$, and so $I\alpha = I\alpha'$ since HG is free, and finally $\alpha = \alpha'$. \square

9 Cofibrations

The first step of our analysis of cofibrations in **Gray-Cat** is to reduce the problem to one about sesquicategories. To this end, we define a sesquifunctor $F : \mathcal{A} \rightarrow \mathcal{B}$ to be a *surjection* if it is surjective on objects and on each hom $F : \mathcal{A}(A, B) \rightarrow \mathcal{B}(FA, FB)$ it is a surjective equivalence (a trivial fibration in **Cat**). We also define a sesquicategory \mathcal{C} to be *projective* if the function $\mathbf{SesquiCat}(\mathcal{C}, F) : \mathbf{SesquiCat}(\mathcal{C}, \mathcal{A}) \rightarrow \mathbf{SesquiCat}(\mathcal{C}, \mathcal{B})$ is surjective for all surjective sesquifunctor $F : \mathcal{A} \rightarrow \mathcal{B}$.

The next result is a direct analogue of [12, Lemma 4.1].

Proposition 9.1 *A Gray-functor $F : \mathcal{A} \rightarrow \mathcal{B}$ is a cofibration if and only if its underlying sesquifunctor $U_*F : U_*\mathcal{A} \rightarrow U_*\mathcal{B}$ has the left lifting property with respect to surjective sesquifunctors. In particular, a Gray-category \mathcal{A} is cofibrant if and only if its underlying sesquicategory $U_*\mathcal{A}$ is projective (with respect to the surjective sesquifunctors).*

PROOF: If $P : \mathcal{C} \rightarrow \mathcal{D}$ is a surjective sesquifunctor then $C_*P : C_*\mathcal{C} \rightarrow C_*\mathcal{D}$ is a trivial fibration of **Gray**-categories. If F is a cofibration then it will have the left lifting property with respect to C_*P , and so U_*F will have the left lifting property with respect to P .

This proves one direction; for the other, suppose that U_*F does have the left lifting property with respect to surjective sesquifunctors. This time, let $P : \mathcal{C} \rightarrow \mathcal{D}$ be a trivial fibration of **Gray**-categories. Then U_*P is clearly surjective, and so U_*F has the left lifting property with respect to U_*P ; but this now means that LU_*F has the left lifting property with respect to P . If now X and Y are **Gray**-functors satisfying $PX = YF$ we have the following diagram of **Gray**-functors

$$\begin{array}{ccccc} LU_*\mathcal{A} & \xrightarrow{I} & \mathcal{A} & \xrightarrow{X} & \mathcal{C} \\ \downarrow LU_*F & & \downarrow F & & \downarrow P \\ LU_*\mathcal{B} & \xrightarrow{J} & \mathcal{B} & \xrightarrow{Y} & \mathcal{D} \end{array}$$

in which I and J denote the canonical inclusions (components of the counit of $L \dashv U_*$). Since LU_*F has the left lifting property with respect to P , there is a **Gray**-functor $Z : LU_*\mathcal{B} \rightarrow \mathcal{C}$ whose composite with LU_*F is XI and whose composite with P is YJ . Now J is bijective on objects, 1-cells, and 2-cells; while P is locally locally fully faithful (bijective on 3-cells with given domains and codomains), and so there is a unique induced $W : \mathcal{B} \rightarrow \mathcal{C}$ with $PW = Y$ and $WJ = Z$. Finally $PWF = YF = PX$ and $WFI = WJ.LU_*F = Z.LU_*F = XI$, while I is bijective on objects, 1-cells, and 2-cells, and P is locally locally fully faithful, and so $WF = X$. Thus X provides the desired fill-in satisfying $WF = X$ and $PW = Y$. \square

For a sesquicategory \mathcal{A} , the counit $HV\mathcal{A} \rightarrow \mathcal{A}$ is bijective on objects, and surjective on 1-cells. Given 1-cells $f, g : x \rightarrow y$ (paths in the 1-cells of \mathcal{A}), a 2-cell in \mathcal{A} between their composites is included in $HV\mathcal{A}$ as one of the generating 2-cells. Thus the counit $E : HV\mathcal{A} \rightarrow \mathcal{A}$ is also full on 2-cells and we have:

Proposition 9.2 *For each sesquicategory \mathcal{A} , the counit map $HV\mathcal{A} \rightarrow \mathcal{A}$ is a surjective sesquifunctor. \square*

We now find ourselves in the typical situation where projectives are the retracts of frees.

Proposition 9.3 *For a sesquicategory \mathcal{A} the following are equivalent:*

- (i) \mathcal{A} is projective;
- (ii) there is a sesquifunctor $J : \mathcal{A} \rightarrow HV\mathcal{A}$ with $EJ = 1$;
- (iii) \mathcal{A} is a retract of a free sesquicategory HG on some computad G .
- (iv) \mathcal{A} is free;

PROOF: The implication (i) \Rightarrow (ii) follows immediately from the fact that E is a surjection of sesquicategories; the implication (ii) \Rightarrow (iii) is trivial, and the implication (iii) \Rightarrow (iv) is Proposition 8.1. For the implication (iv) \Rightarrow (i) we must show that any free sesquicategory HG is projective. But to say that HG is projective with respect to a surjective sesquifunctor $P : \mathcal{A} \rightarrow \mathcal{B}$ is equivalent to saying that G is projective with respect to the underlying computad morphism $VP : V\mathcal{A} \rightarrow V\mathcal{B}$. But for a surjective sesquifunctor $P : \mathcal{A} \rightarrow \mathcal{B}$, clearly $VP : V\mathcal{A} \rightarrow V\mathcal{B}$ has a section, and so any object G will be projective with respect to VP . \square

Corollary 9.4 *A **Gray**-category is cofibrant if and only if its underlying sesquicategory is free on a computad.* \square

We can now give a very explicit description of a cofibrant replacement functor. For a **Gray**-category \mathbb{A} , first forget the 3-cells, to obtain a sesquicategory $U_*\mathbb{A}$, then form the free sesquicategory $HVU_*\mathbb{A}$ on its underlying computad, and then the free **Gray**-category $LHVU_*\mathbb{A}$ on that. This has a canonical **Gray**-functor $E' : LHVU_*\mathbb{A} \rightarrow \mathbb{A}$ to the original **Gray**-category; in fact this is the counit at \mathbb{A} of an adjunction between **Gray**-categories and computads. This **Gray**-functor E' is bijective on objects, surjective on 1-cells, and full on 2-cells. Factorize it as

$$LHVU_*\mathbb{A} \xrightarrow{i} Q\mathbb{A} \xrightarrow{q} \mathbb{A}$$

where i is bijective on objects, 1-cells, and 2-cells, and q is fully faithful on 3-cells. Clearly this defines a functor $Q : \mathbf{Gray-Cat} \rightarrow \mathbf{Gray-Cat}$ and a natural transformation $q : Q \rightarrow 1$.

Since i is bijective on objects, 1-cells, and 2-cells, and the underlying sesquicategory of $LHVU_*\mathbb{A}$ is just the free sesquicategory $HVU_*\mathbb{A}$; the underlying sesquicategory of $Q\mathbb{A}$ is also just $HVU_*\mathbb{A}$, so $Q\mathbb{A}$ is indeed a cofibrant **Gray**-category. Furthermore, since E' is bijective on objects, surjective on 1-cells, and full on 2-cells, so is q ; since q is also fully faithful on 3-cells, it is in fact a trivial fibration of **Gray**-categories.

But in fact we can do a bit better: Q is actually a comonad, so we can obtain a category of “weak” morphisms of **Gray**-categories by taking the Kleisli category. This turns out to be a special case of a general construction due to Garner [3].

We already have the counit $q : Q \rightarrow 1$; next we build the comultiplication $d : Q \rightarrow Q^2$. To do this, write $W : \mathbf{Gray-Cat} \rightarrow \mathbf{Cmptd}$ for the composite of $U_* : \mathbf{Gray-Cat} \rightarrow \mathbf{SesquiCat}$ and $V : \mathbf{SesquiCat} \rightarrow \mathbf{Cmptd}$, and K for the left adjoint HV ; with unit $\eta : 1 \rightarrow VK$ and counit $\epsilon : KV \rightarrow 1$. Observe that the exterior of

$$\begin{array}{ccccc}
 & & FU\mathbb{A} & & \\
 & F\eta U\mathbb{A} \swarrow & & \searrow 1 & \\
 FU FU\mathbb{A} & \xrightarrow{\epsilon FU\mathbb{A}} & FU\mathbb{A} & & \\
 \downarrow FU i\mathbb{A} & & \downarrow i\mathbb{A} & & \\
 FU Q\mathbb{A} & \xrightarrow{\epsilon Q\mathbb{A}} & Q\mathbb{A} & & \\
 \downarrow i Q\mathbb{A} & & \downarrow q Q\mathbb{A} & & \\
 & & QQ\mathbb{A} & &
 \end{array}$$

commutes, and that $i\mathbb{A}$ is bijective on objects, 1-cells, and 2-cells, while $qQ\mathbb{A}$ is fully faithful on 3-cells. Thus

there is a unique **Gray**-functor $d : Q\mathbb{A} \rightarrow Q^2\mathbb{A}$ making the diagram

$$\begin{array}{ccc}
FU\mathbb{A} & \xrightarrow{i\mathbb{A}} & Q\mathbb{A} \\
F\eta U\mathbb{A} \downarrow & & \downarrow 1 \\
FU FU\mathbb{A} & & \\
FU i\mathbb{A} \downarrow & d\mathbb{A} \nearrow & \\
FU Q\mathbb{A} & & \\
iQ\mathbb{A} \downarrow & & \\
Q^2\mathbb{A} & \xrightarrow{qQ\mathbb{A}} & Q\mathbb{A}
\end{array}$$

commute. These $d\mathbb{A}$ are the components of a natural transformation $d : Q \rightarrow Q^2$; naturality of d is inherited from that of η , ϵ , i , and q . One of the counit laws $qQ.d = 1$ holds by definition of d . We prove the other law $Qq.d = 1$, by checking that the composite of each side with i agrees (so that $Qq.d$ and 1 agree on objects, 1-cells, and 2-cells), and that the composite of each side with q agrees (so that the two sides agree on 3-cells). The calculations are:

$$\begin{aligned}
q.Qq.d &= q.qQ.d \\
&= q.1 \\
Qq.d.i &= Qq.iQ.FUi.F\eta U \\
&= i.Fuq.FUi.F\eta U \\
&= i.FU\epsilon.F\eta U \\
&= i \\
&= 1.i
\end{aligned}$$

and so $Qq.d = 1$. We prove the coassociative law $Qd.d = dQ.d$ using the same technique:

$$\begin{aligned}
qQ^2.Qd.d &= d.qQ.d \\
&= d \\
&= qQ^2.dQ.d \\
Qd.d.i &= Qd.iQ.FUi.F\eta U \\
&= iQ^2.FUd.FUi.F\eta U \\
&= iQ^2.FUiQ.FU FU i.FU F\eta U.F\eta U \\
&= iQ^2.FUiQ.F\eta U Q.FUi.F\eta U \\
&= dQ.iQ.FUi.F\eta U \\
&= dQ.d.i
\end{aligned}$$

and so $Qd.d = dQ.d$.

We have now proved:

Theorem 9.5 *There is a comonad (Q, q, d) on **Gray-Cat** for which $q\mathbb{A} : Q\mathbb{A} \rightarrow \mathbb{A}$ is a cofibrant replacement of \mathbb{A} for every **Gray**-category \mathbb{A} .*

As mentioned above, this comonad can be obtained using the results of [3], which provide a general technique for defining weak morphisms of higher categories via Kleisli categories for comonads.

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